The Papkovich-Neuber representation

Any body-force-free Stokes flow can be written in the form

$$\boxed{2\mu \boldsymbol{u} = \nabla\left(\boldsymbol{x}\cdot\boldsymbol{\Phi}+\chi\right) - 2\boldsymbol{\Phi} \quad \text{with} \quad p = \nabla\cdot\boldsymbol{\Phi}}$$

where χ is a harmonic scalar and $\boldsymbol{\Phi}$ is a harmonic vector. This result will have been quoted, but probably not derived, in lectures – it's extremely powerful and really useful in lots of different problems, so it's quite neat to have a bit of an idea how it comes about.

Start by letting $p = \nabla^2\Pi$ for some function Π. Thinking back to IB Methods, this can always be done (except for probably in some weird pure maths edge cases that we'd never see in physics) by setting

$$\Pi = -\frac{1}{4\pi}\int_V \frac{p\left(\boldsymbol{x'}\right)}{\left|\boldsymbol{x}-\boldsymbol{x'}\right|}\,\mathrm{d}V.$$

Thus we can rewrite the Stokes equations as $\nabla^2\left(\mu\boldsymbol{u}-\nabla\Pi\right)=0$ (plus $\nabla\cdot\boldsymbol{u}=0$). This means that

$$\mu\boldsymbol{u} = \nabla\Pi - \boldsymbol{\Phi} \quad \text{where } \nabla^2\boldsymbol{\Phi}=0.$$

Further imposing incompressibility, $\nabla^2\Pi = \nabla\cdot\boldsymbol{\Phi}$, and so $p=\nabla\cdot\boldsymbol{\Phi}$. Then, since Π satisfies this Poisson equation,

$$\Pi = \frac{1}{2}\left(\boldsymbol{x}\cdot\boldsymbol{\Phi}+\chi\right),$$

with $\nabla^2\chi = 0$ (it's easy to check that this is the general solution). Thus, rearranging and substituting in our form for Π, we get the desired Papkovich-Neuber representation. There are some easy harmonic trial functions to use when constructing Φ and χ – if we want the solution to decay as $r\to\infty$, try

$$\frac{1}{r},\ \nabla\left(\frac{1}{r}\right), \nabla\nabla\left(\frac{1}{r}\right), \ldots$$

Alternatively, if we want the solution to be regular as $r\to 0$, try

$$1,\ r^3\nabla\left(\frac{1}{r}\right),\ r^5\nabla\nabla\left(\frac{1}{r}\right), \ldots$$

Remember as well that $\boldsymbol{u}$, $\boldsymbol{F}$, $\boldsymbol{x}$ are all 'true' quantities and $\boldsymbol{\Omega}$, $\boldsymbol{G}$, $\boldsymbol{\omega}$ are all 'pseudo'. Less rigorously, but helpful as a memory aid, ∇ is 'true' but $\wedge$ (i.e. the cross product) is 'psuedo'. Thus $\nabla\wedge\boldsymbol{x}$ is 'pseudo' but $\nabla\wedge\boldsymbol{\omega}$ is 'true' (pseudo $\times$ psuedo = true) – the two types of tensor obey a 'parity-like' relationship.

Stokeslet solution

One of the simplest solutions we can derive here is that of the flow due to a point force at the origin in three dimensions. This is incredibly useful as we can sum these solutions together (see Part III Slow Viscous Flow). We're not breaking the rule of 'body-force-free' here, since the body force is only supported at the origin ($\boldsymbol{f} = \boldsymbol{F}\delta\left(\boldsymbol{x}\right)$). Try $\chi = 0$ and $\boldsymbol{\Phi} = \alpha\boldsymbol{F}/r$; after a bit of algebra, this gives

$$2\mu\boldsymbol{u} = -\alpha\left\{\frac{\boldsymbol{F}}{r}+\frac{\left(\boldsymbol{F}\cdot\boldsymbol{x}\right)\boldsymbol{x}}{r^3}\right\} \Rightarrow \underline{\underline{\boldsymbol{\sigma}}} = \frac{3\alpha\left(\boldsymbol{F}\cdot\boldsymbol{x}\right)}{r^5}\boldsymbol{x}\boldsymbol{x}.$$

It then remains to integrate the surface stress over a sphere of radius R centred on the origin and match with force to determine the constant α. To do this, we must note that $\int_{r=R} n_i n_j\,\mathrm{d}S$ is an isotropic integral, and so must equal $K\delta_{ij}$ for some constant K (throwback to Part IA Vector Calc here). Setting $i=j$, $4\pi R^2 = 3K$ and so $K = 4\pi R^2/3$. Thus,

$$\int_{r=R}\underline{\underline{\boldsymbol{\sigma}}}\cdot\boldsymbol{n}\,\mathrm{d}S = \frac{3\alpha\boldsymbol{F}}{R^2}\cdot\int_{r=R}\boldsymbol{n}\boldsymbol{n}\,\mathrm{d}S = 4\pi\alpha\boldsymbol{F}.$$

But $\nabla\cdot\underline{\underline{\sigma}} = -\boldsymbol{F}\delta\left(\boldsymbol{x}\right)$, $\int_V \boldsymbol{F}\delta\left(\boldsymbol{x}\right)\,\mathrm{d}V = \boldsymbol{F} = -\int_{r=R}\underline{\underline{\boldsymbol{\sigma}}}\cdot\boldsymbol{n}\,\mathrm{d}S$. Hence $\alpha = -1/4\pi$.

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