Kelvin's circulation theorem

Question 7 on example sheet 2 throws up a few topics about conservation of angular momentum (vortex stretching, the 'ballerina effect') with a flow that looks a bit like a tornado. There's a really neat way of looking at this using Kelvin's circulation theorem, a starred section of the IB Course that used to be (back in my day, and definitely back in the day that this question was first set) examinable. Here I'll outline the theorem and show how you can apply it to the flow in this question.

The theorem and its 'proof'

Circulation is defined around a closed contour γ by the integral expression

$$\Gamma = \oint_\gamma \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x},$$

where $\mathrm{d}\boldsymbol{x}$ is the (tangent) line element. Provided that the density is constant and body forces are conservative ($\boldsymbol{F} = -\boldsymbol{\nabla}\chi$), Kelvin's circulation theorem states that the material derivative of Γ with respect to time around a material fluid curve (i.e. a γ that moves with the flow) is zero. This condition can be expressed as

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0 \quad \text{where} \quad \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}.$$

In order to prove this result, note that

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_\gamma \left[\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{x} + \boldsymbol{u} \cdot \frac{\mathrm{D}\left(\mathrm{d}\boldsymbol{x}\right)}{\mathrm{D}t}\right],$$

and the curve γ is deforming with the flow, so $\mathrm{D}\left(\mathrm{d}\boldsymbol{x}\right)/\mathrm{D}t$ is just $\mathrm{d}\boldsymbol{u}$. Then, substituting from the Euler equation,

$$\begin{aligned}
\frac{\mathrm{D}\Gamma}{\mathrm{D}t} &= \oint_\gamma \left[-\boldsymbol{\nabla}\left(\frac{p}{\rho} + \chi\right) \cdot \mathrm{d}\boldsymbol{x} + \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{u}\right], \\
&= \oint_\gamma \left[-\boldsymbol{\nabla}\left(\frac{p}{\rho} + \chi\right) \cdot \mathrm{d}\boldsymbol{x} + \mathrm{d}\left(\frac{1}{2}\left|\boldsymbol{u}\right|^2\right)\right], \\
&= \oint_\gamma \boldsymbol{\nabla}\left(\frac{1}{2}\left|\boldsymbol{u}\right|^2 - \frac{p}{\rho} - \chi\right) \cdot \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

This integral must be zero, since it is the integral of the gradient of a single-valued function around a closed curve (no weird winding numbers/branch cut stuff here!), and so $\mathrm{D}\Gamma/\mathrm{D}t = 0$ as desired.

Applying this to sheet 2 Q7

The tornado-like flow in question 7 provides a good example to illustrate the use of Kelvin's circulation theorem, where we start by writing the flow in cylindrical polar coordinates $(u_r,\, u_\theta,\, u_z)$ as

$$\boldsymbol{u} = (-\alpha r,\, r^2 f(t),\, 2\alpha z).$$

These three components illustrate the radial inwards flow, the rotation around the z axis and the movement along the axis, respectively, and it is clear that rotation must happen faster as the flow is dragged in towards $r = 0$ so as to conserve angular momentum. Compute the circulation around a material curve γ defined to be the circle $r = R(t)$, which remains a circle for all time, with the radius shrinking,

$$\Gamma = \oint_{\{r=R(t)\}} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = R(t) \int_0^{2\pi} u_\theta \, \mathrm{d}\theta = 2\pi R(t)^3 f(t).$$

(Notice that this integral could also be computed using Stokes' theorem and your expression for $\boldsymbol{\omega}$ from the start of the question – why not check that the answers match?) On the material curve $r = R(t)$, this value of Γ is constant, so it remains to find how $R(t)$ evolves in time to deduce how $f(t)$ therefore must change. Since this is a material curve,

$$\frac{\mathrm{d}R}{\mathrm{d}t} = u_r|_{r=R(t)} = -\alpha R(t),$$

and so $R(t) = R_0 e^{-\alpha t}$ for some constant R_0. Thus,

$$2\pi R_0^3 e^{-3\alpha t} f(t) = \text{const.},$$

and therefore $f(t) \propto e^{3\alpha t}$, which can also be seen by considering the contributions to the vorticity equation (the 2α seen from the example in lectures plus an extra α contribution from rotation).

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