The dynamic boundary condition for water waves

This is something that a lot of you have asked about, or missed some subtle points, and I realise that it's also something I perhaps haven't explained as clearly as possible in the supervisions. Here I'll work through perhaps the most comprehensive problem you could ever be expected to solve in an exam, incorporating some surface tension too just to make it especially messy (and interesting). You won't be expected to recall what effect surface tension has on the boundary conditions – if they were to ask about it (unlikely, but I've seen it in the past), it'll be given.

Problem: *deduce the dispersion relation for water waves (of the form $z = \eta(x, t) = \eta_0 \mathrm{Re}\left[e^{\mathrm{i}(kx-\omega t)}\right]$) at the interface $z = 0$ between two fluid layers of semi-infinite depth, the lower with a density ρ_2 and the upper with density ρ_1. There is surface tension at the interface between the waves which has the effect of introducing a jump condition in p across the interface, $p|_-^+ = \gamma \partial^2 \eta / \partial x^2$* [1].

Start, as always, by introducing velocity potentials for the irrotational flow above and below the interface, namely

$$\boldsymbol{u} = \nabla \phi_1 \text{ in } z > \eta(x, t) \text{ and } \boldsymbol{u} = \nabla \phi_2 \text{ in } z < \eta(x, t). \tag{1}$$

So we can match later on, these are going to have to have the form $\phi_i = \tilde{\phi}_i(z)\,\mathrm{Re}\left[e^{\mathrm{i}(kx-\omega t)}\right]$. Implicitly, drop the 'real part' notation as we will always assume that this is the case. By incompressibility, $\nabla^2 \phi_i = 0$ and so we must solve

$$\tilde{\phi}_i'' - k^2 \tilde{\phi}_i = 0. \tag{2}$$

This has solutions of the form $\tilde{\phi}_i \propto e^{\pm kz}$, so take

$$\tilde{\phi}_1 = \Phi_1 e^{-kz} \quad \text{and} \quad \tilde{\phi}_2 = \Phi_2 e^{kz}, \tag{3}$$

through requiring decay at $\pm\infty$. From hereon in, drop the tildes from the functions for clarity.

The kinematic boundary condition at the interface implies that

$$\frac{D\eta}{Dt} = \frac{\partial \phi_1}{\partial z} e^{\mathrm{i}(kx-\omega t)} = \frac{\partial \phi_2}{\partial z} e^{\mathrm{i}(kx-\omega t)} \quad \text{on } z = \eta(x, t), \tag{4}$$

i.e. the vertical velocities match at the boundary and are equal to the vertical velocity of the boundary itself. If we assume $|\eta| \ll 1$, we can linearise onto $z = 0$ and this condition becomes

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial z} e^{\mathrm{i}(kx-\omega t)} = \frac{\partial \phi_2}{\partial z} e^{\mathrm{i}(kx-\omega t)} \quad \text{on } z = 0. \tag{5}$$

This condition becomes

$$-\mathrm{i}\omega\eta_0 = -k\Phi_1 = k\Phi_2. \tag{6}$$

The unsteady Bernoulli theorem provides a dynamic boundary condition for this problem. Therefore,

$$\rho_i \frac{\partial}{\partial t}\left[\phi_i(z)\, e^{\mathrm{i}(kx-\omega t)}\right] + \frac{\rho_i}{2}|\boldsymbol{u}_i|^2 + p_i + \rho g \eta(x, t) = f_i(t) \tag{7}$$

which we again linearise around onto $z = 0$, only keeping terms of first order, so

$$\rho_i \frac{\partial}{\partial t}\left[\phi_i(z)\, e^{\mathrm{i}(kx-\omega t)}\right] + p_i + \rho_i g \eta_0 e^{\mathrm{i}(kx-\omega t)} = f_i(t). \tag{8}$$

By requiring no x-dependence in the function on the left-hand side, and expecting the pressure too to have the form $\propto e^{\mathrm{i}(kx-\omega t)}$, it is clear that $f_1 = f_2 = 0$. **It's really important that you leave the pressures in here** – this is one place where you can't take them to be wlog zero! The dynamic boundary condition is essentially all to do with pressures across an interface, so take care – even when the pressure is continuous across the interface, it may be nonzero, so don't just get rid of it. Only once we've realised that $f_1 = f_2 = 0$ can we equate the expressions either side of the boundary,

$$-\mathrm{i}\omega\rho_1\Phi_1 + P_1 + \rho_1 g\eta_0 = -\mathrm{i}\omega\rho_2\Phi_2 + P_2 + \rho_2 g\eta_0, \tag{9}$$

where $p_i = P_i e^{\mathrm{i}(kx-\omega t)}$. The pressure boundary condition gives

$$P_1 - P_2 = -\gamma k^2 \eta_0. \tag{10}$$

[1] Only valid for $|\eta_x| \ll 1$, which (a) is the case here and (b) isn't something you should be worrying about.

j.webber@damtp.cam.ac.uk

Substituting in for Φ_1 and Φ_2 from equation (6) implies that

$$\begin{aligned} \frac{\omega^2 \rho_1 \eta_0}{k} + \rho_1 g \eta_0 + \frac{\omega^2 \rho_2 \eta_0}{k} - \rho_2 g \eta_0 &= P_2 - P_1 \\ &= \gamma k^2 \eta_0, \end{aligned} \tag{11}$$

and therefore

$$\omega^2 = \frac{(\rho_2 - \rho_1)\, gk + \gamma k^3}{\rho_1 + \rho_2}. \tag{12}$$

Let's just sanity-check this answer – there's a few comments to be made:

- Firstly, how about if there's no surface tension effects? Then $\omega^2 = gk\,(\rho_2 - \rho_1)\,/\,(\rho_1 + \rho_2)$. This should be a dispersion relation you recognise, and all looks fine!
- Secondly, what about $\gamma = 0$ **and** $\rho_1 = \rho_2$? Then $\omega = 0$. We can't get waves when there isn't an interface – since this is the case of just drawing a line in the middle of one big fluid!
- Finally, how about the case of an instability (this is very much not on the IB course!), where $\rho_1 > \rho_2$ – in this case we'd expect the heavier fluid to fall down and the lighter fluid to rise up, and instead of waves we'd get exponentially growing modes along the interface. Let $\omega = \mathrm{i}\sigma$, where σ is the growth rate (i.e. $\eta \sim e^{\sigma t}$) and our relation becomes

$$\sigma^2 = \frac{(\rho_1 - \rho_2)\, gk - \gamma k^3}{\rho_1 + \rho_2}; \tag{13}$$

so $\rho_1 \gg \rho_2$ implies faster growth, as we'd expect, and the surface tension acts to stabilise the interface, as we'd also expect. If there's some tension in the interface, it won't want to grow, and actually, very large γ can support a density profile that's 'the wrong way up'.

 j.webber@damtp.cam.ac.uk