Applied Mathematics Seminar
University of Nottingham, Thursday 15th January 2026, 15:00

# A linear-elastic-nonlinear-swelling model for hydrogels

**Joseph Webber** and Tom Montenegro-Johnson
Mathematics Institute, University of Warwick

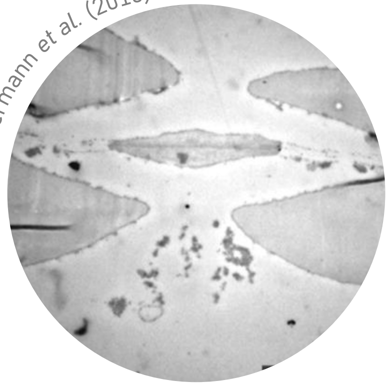
Grae Worster
DAMTP, University of Cambridge

# Modelling hydrogels

Formed of a hydrophilic polymer scaffold surrounded by adsorbed water molecules

- can comprise >99% water by volume but remain solid
- behave elastically with low shear modulus
- swell or dry to extreme degrees when water is added or removed

Royal Horticultural Society



Pittermann *et al.* (2010)


*Final radius of ~1.5cm*

# Hydrogels

## Fully-nonlinear models

$$W = W_{\text{mix}} + W_{\text{elastic}}$$

- Energy density function with contributions from mixing (entropy, electrostatic interactions, temperature-dependence, …) and elasticity (of individual polymer chains).
- Accurate, models large strains
- Not analytically tractable, parameters hard to determine

*Flory & Rehner (1943a,b), Cai & Suo (2012), Bertrand et al. (2016), Butler & Montenegro-Johnson (2022)*

## Fully-linear models

$$\frac{\partial \phi}{\partial t} = D\frac{\partial^2 \phi}{\partial x^2} \qquad \left(D = K + \frac{4}{3}\mu\right)$$

- Based on linear poroelasticity, interstitial flow via Darcy's law. Treats gel as a linear-elastic material.
- Analytically tractable, clear physics, 'macroscopic' parameters
- Can't deal with large swelling strain

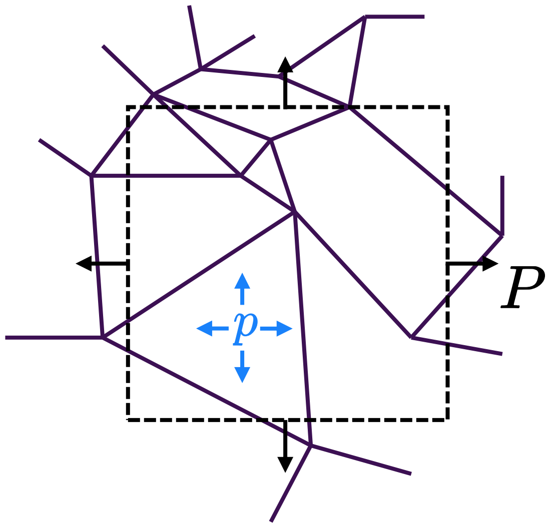
*Biot (1941), Tanaka & Fillmore (1979), Doi (2009)*

# Poromechanics

**A geophysicist's approach:** separate contributions from stress into a 'pore pressure' and an 'effective stress'

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}_{\text{eff}}$$

**Bulk pressure (or thermodynamic pressure)** the isotropic stress exerted by a sample of gel; our familiar concept of pressure




**Pervadic pressure (or Darcy pressure, "pore" pressure)** is the pressure as would be measured by a transducer separated by a partially-permeable membrane from the gel.

**In soil science:** *p* is the pore pressure, *P* is the overburden pressure
**In colloids:** *p* is [related to] the chemical potential, *Π* is the osmotic pressure

$$P = p + \Pi$$

*osmotic effects?*
*isotropic elasticity?*
**generalised** osmotic pressure

*p* drives flows by Darcy's law:

$$\boldsymbol{u} = -\frac{k}{\mu_l}\boldsymbol{\nabla} p$$

permeability

relative fluid flux
$\boldsymbol{u} = (1-\phi)(\boldsymbol{u_w} - \boldsymbol{u_p})$

(dynamic) viscosity of fluid

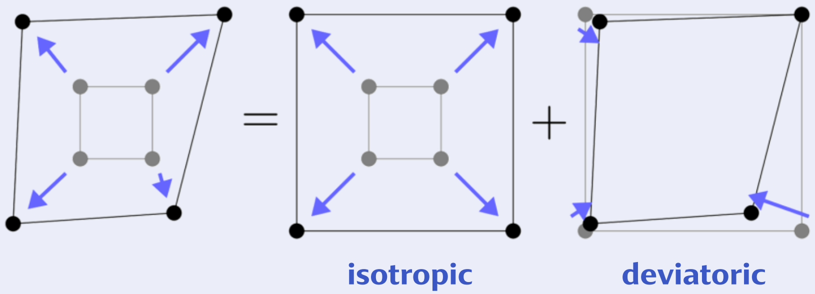
Biot (1941) *J. Appl. Phys.* ***12**:155–164*; Peppin *et al.* (2004) *Phys. Fluids* ***17**:053301*

# Poromechanics

**Linear (Biot) poroelasticity** specifies a linear-elastic constitutive relation linking strains to effective stresses. Hydrogels swell a lot, with potentially large strains: linear is no good!

One way around this: use **finite strain (nonlinear) elastic models** for effective stress.

*e.g. Hencky model* $$\boldsymbol{\sigma}_{\text{eff}} = \frac{\Lambda\phi}{2}\,\mathrm{tr}\big(\ln(\mathbf{F}\mathbf{F}^{\mathrm{T}})\big)\mathbf{I} + \frac{M-\Lambda}{2}\ln(\mathbf{F}\mathbf{F}^{\mathrm{T}})$$




assume linearity only in the **deviatoric strain** from some **fully-swollen reference state**

"linear-elastic materials with properties dependent on swelling state"

Biot (1941) *J. Appl. Phys.* ***12:****155-164*; Tanaka *et al.* (1973) *J. Chem Phys.* ***59:****5151-5159*; MacMinn *et al.* (2016) *Phys. Rev. Appl.* ***5:****044020*

# The linear-elastic-nonlinear-swelling model

Therefore, the deviatoric part of the effective stress tensor must depend linearly on the deviatoric part of the Cauchy strain (the isotropic part could be huge)

$$\mathbf{e} = \frac{1}{2}\left[(\boldsymbol{\nabla}\boldsymbol{\xi}) + (\boldsymbol{\nabla}\boldsymbol{\xi})^{\mathsf{T}}\right] = \left[1 - \left(\frac{\phi}{\phi_0}\right)^{1/3}\right]\mathbf{I} + \boldsymbol{\epsilon}$$

**isotropic strain**
depends only on degree to which gel is swollen

**deviatoric strain**
assumed small

This allows us to construct a stress tensor like usual

$$\boldsymbol{\sigma}_{\text{eff}} = -\Pi(\phi)\mathbf{I} + 2\mu_s(\phi)\boldsymbol{\epsilon}$$

**shear modulus depends on swelling!**
dry gels will probably be stiffer

**isotropic part must be (-) osmotic pressure**
since the isotropic part of the total stress tensor is (-) the bulk pressure

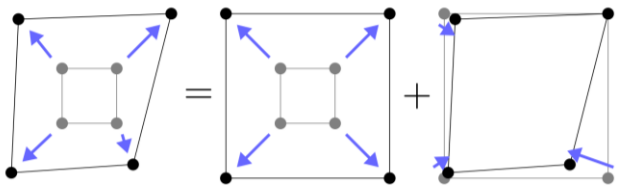
**depends on swelling alone**
isotropic strains lead to isotropic stresses – see the isotropic part of strain tensor

physically intuitive result

JJW & Worster (2023) *J. Fluid Mech.* ***960****:A37*

# The linear-elastic-nonlinear-swelling model

$$\boldsymbol{\sigma} = -p\mathbf{I} + \boldsymbol{\sigma}_{\text{eff}}$$
$$P = p + \Pi$$




$$\mathbf{e} = \frac{1}{2}\left[(\boldsymbol{\nabla}\boldsymbol{\xi}) + (\boldsymbol{\nabla}\boldsymbol{\xi})^{\mathsf{T}}\right] = \left[1 - \left(\frac{\phi}{\phi_0}\right)^{1/3}\right]\mathbf{I} + \boldsymbol{\epsilon}$$

$$\boldsymbol{\sigma}_{\text{eff}} = -\Pi(\phi)\mathbf{I} + 2\mu_s(\phi)\boldsymbol{\epsilon}$$
$$\boldsymbol{u} = (1-\phi)(\boldsymbol{u_w} - \boldsymbol{u_p})$$
$$\boldsymbol{q} = (1-\phi)\boldsymbol{u_w} + \phi\boldsymbol{u_p}$$

- Have an expression for stress in the gel, so conservation of momentum links pressure gradients to deviatoric strains,

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} = 0 \quad \text{so} \quad \underbrace{\boldsymbol{\nabla}p = -\boldsymbol{\nabla}\Pi(\phi)}_{\text{pervadic pressure gradients oppose osmotic ones}} + 2\boldsymbol{\nabla}\cdot\left[\mu_s(\phi)\boldsymbol{\epsilon}\right]$$

- Since gradients in pervadic pressure drive flows, this allows us to describe gel reconfiguration (when coupled with conservation of polymer and water)

$$\frac{\partial\phi}{\partial t} + \boldsymbol{q}\cdot\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\cdot(\phi\boldsymbol{u}) \quad \text{alongside} \quad \boldsymbol{u} = -\frac{k(\phi)}{\mu_l}\boldsymbol{\nabla}p$$

depends on swelling ($k(\phi)$)

phase-averaged (gel and water) flux ($\boldsymbol{q}$)

$$\boxed{\frac{\partial\phi}{\partial t} + \boldsymbol{q}\cdot\boldsymbol{\nabla}\phi = \boldsymbol{\nabla}\cdot\left\{\frac{k(\phi)}{\mu_l}\left[\phi\frac{\partial\Pi}{\partial\phi} + \frac{4\mu_s(\phi)}{3}\left(\frac{\phi}{\phi_0}\right)^{1/3}\right]\boldsymbol{\nabla}\phi\right\}}$$

JJW & Worster (2023) *J. Fluid Mech.* ***960****:A37*; JJW et al. (2023) *J. Fluid Mech.* ***960****:A38*

# Characterising a gel

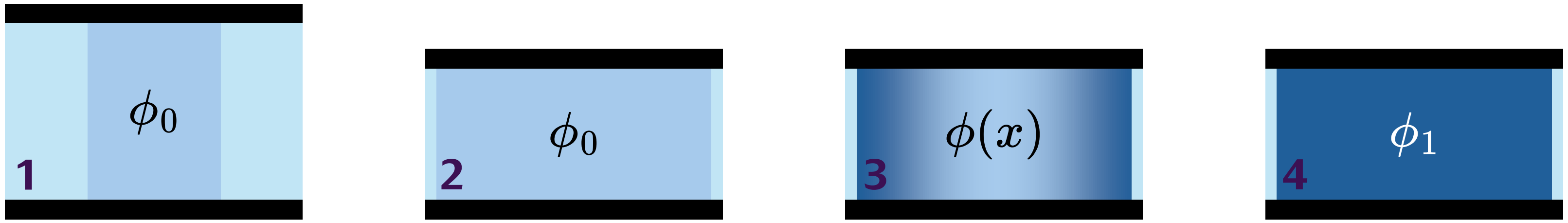
$$\frac{\partial \phi}{\partial t} + \boldsymbol{q} \cdot \boldsymbol{\nabla} \phi = \boldsymbol{\nabla} \cdot \left\{ \frac{k(\phi)}{\mu_l} \left[ \phi \frac{\partial \Pi}{\partial \phi} + \frac{4\mu_s(\phi)}{3} \left( \frac{\phi}{\phi_0} \right)^{1/3} \right] \boldsymbol{\nabla} \phi \right\}$$

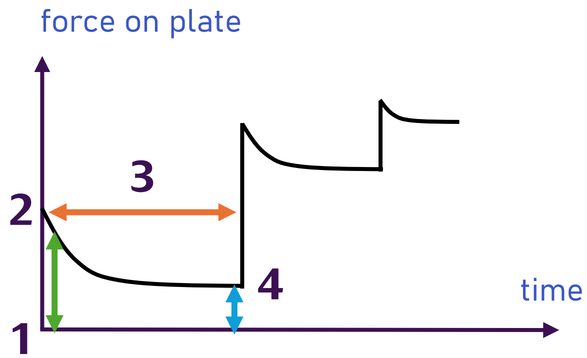
$$\boldsymbol{\sigma} = -[p + \Pi(\phi)]\mathsf{I} + 2\mu_s(\phi)\boldsymbol{\epsilon} \qquad \boldsymbol{u} = \frac{k(\phi)}{\mu_l} \left[ \frac{\partial \Pi}{\partial \phi} + \frac{4\mu_s(\phi)}{3\phi} \left( \frac{\phi}{\phi_0} \right)^{1/3} \right] \boldsymbol{\nabla} \phi$$

**Shear modulus** characterises the stiffness of a hydrogel and describes the initial elastic response before water diffuses through the structure

**Osmotic pressure** characterises the affinity for water ('desire' to swell or deswell)

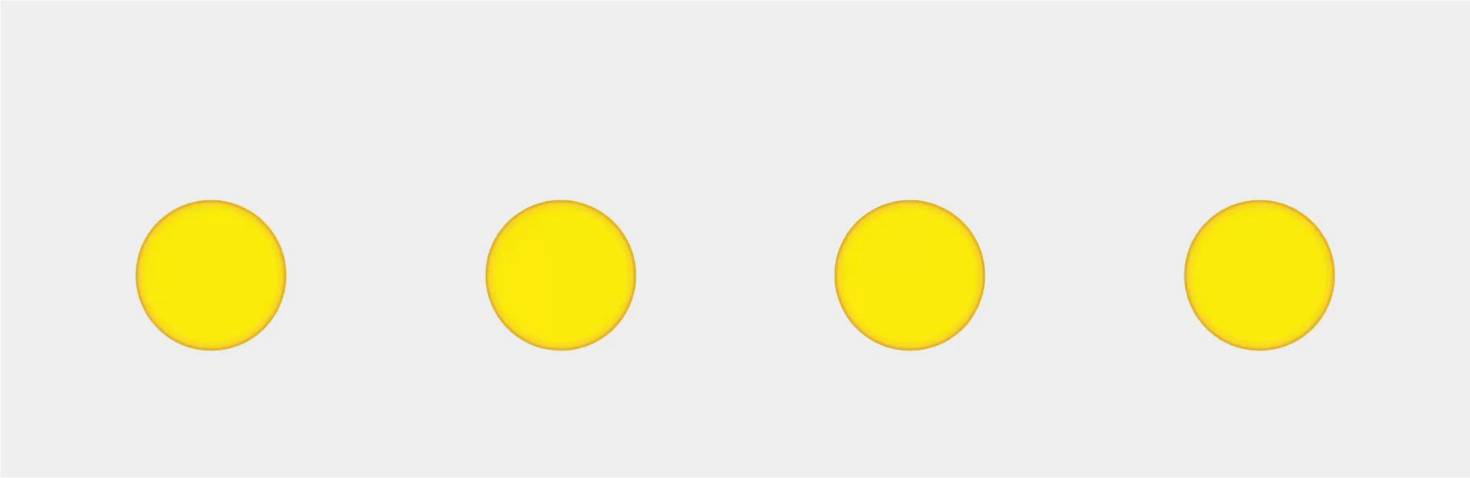
**Permeability** describes the resistance to viscous flow through the pore scaffold




JJW & Worster (2023) *J. Fluid Mech.* ***960**:A37*; JJW et al. (2023) *J. Fluid Mech.* ***960**:A38*; JJW (2024) *PhD thesis, U. Cambridge*

# Insights from LENS modelling



Uniaxial problems are easy: shape set by polymer conservation

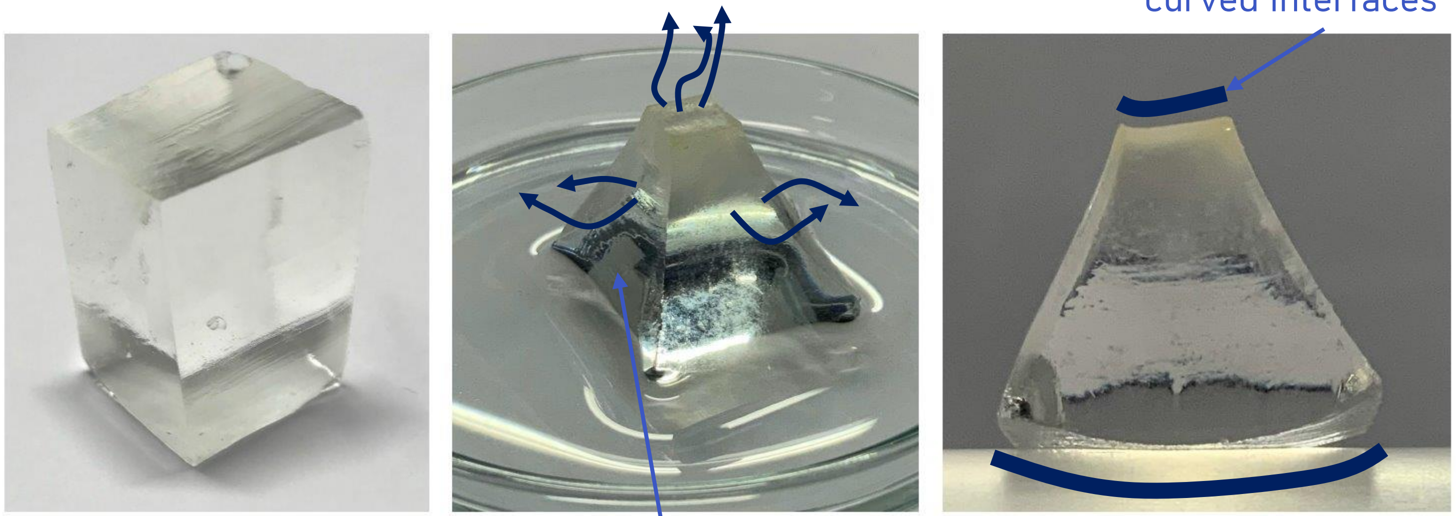
1+1 dimensional advection-diffusion equation

BC at gel-water interface set by stress and pressure balance

More complex geometries = hard!

Need to relate swelling state to displacement to find how the shape evolves

$$\nabla^4 \boldsymbol{\xi} = -3\boldsymbol{\nabla}\nabla^2(\phi/\phi_0)^{1/3}$$

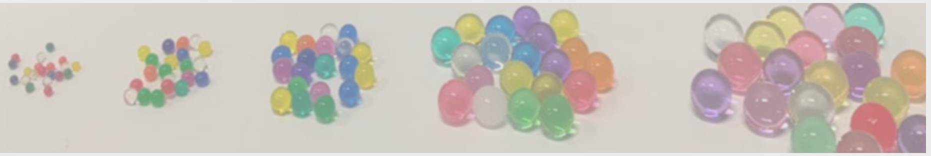



JJW & Worster (2023) *J. Fluid Mech.* ***960****:A37*; JJW et al. (2023) *J. Fluid Mech.* ***960****:A38*

# Overview

## 1. LENS modelling of gels

Osmosis · elastic stresses · equilibrium swelling · transport of water · shape change in swelling/drying · behaviour at interfaces · characterising a gel

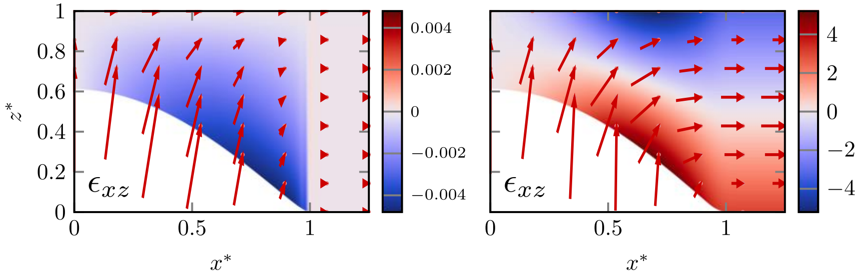


*Temperature (representative values)*

270 K — 300 K — 310 K

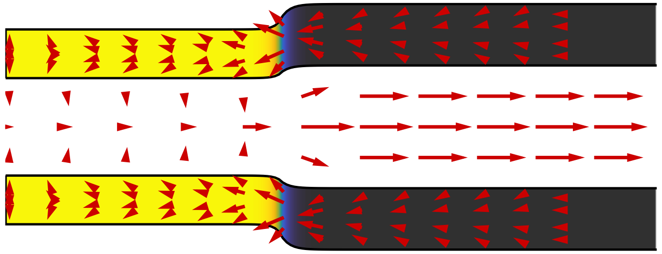
## 2. Freezing gels at low*(-ish)* temperatures

Formation of pure ice · cryosuction · applying model to GelFrO · a 2D model · stress buildup

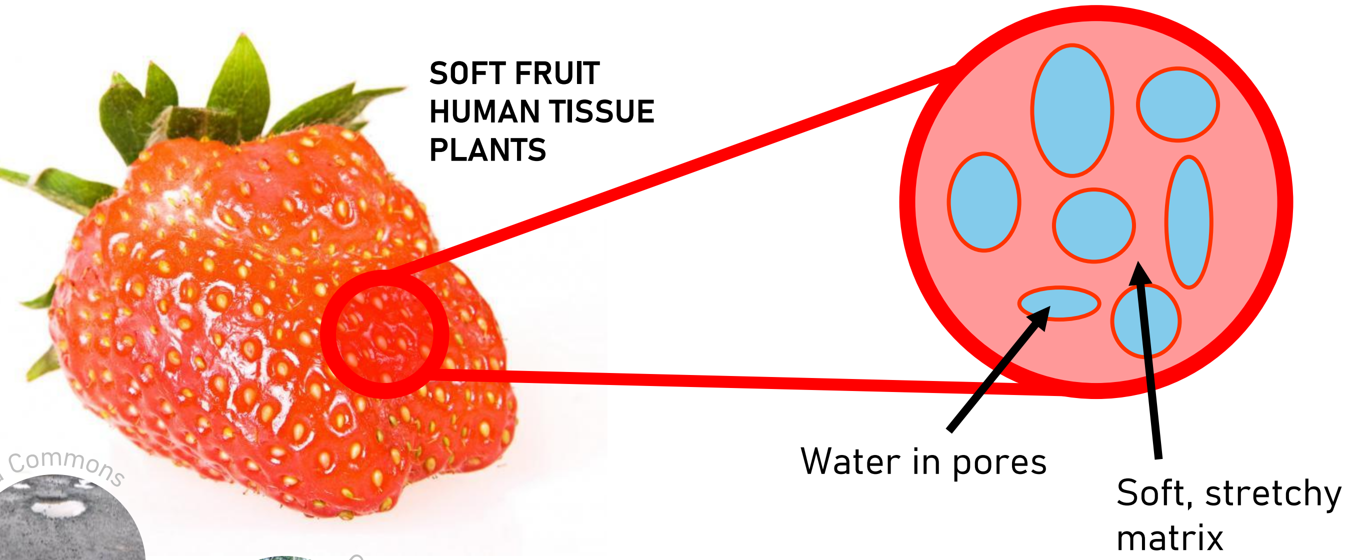



## 3. Building with thermo-responsive gels

Heating effects on swelling · heat transfer in gels · building pumps with collapsing tubes



# How do deformable porous media freeze?





TARMAC


SOIL

$$(L/L_0)^3 = V/V_0 \approx 1.09$$

$$\epsilon = (L - L_0)/L_0 \approx 0.02$$

stresses must therefore scale like *0.02E*

**but,** for a strawberry, *E ~ $10^5$* Pa yet the fracture strength ~ *2 × $10^4$* Pa

**An *et al.*** Food Res. Int. **169:**112787 (2023)

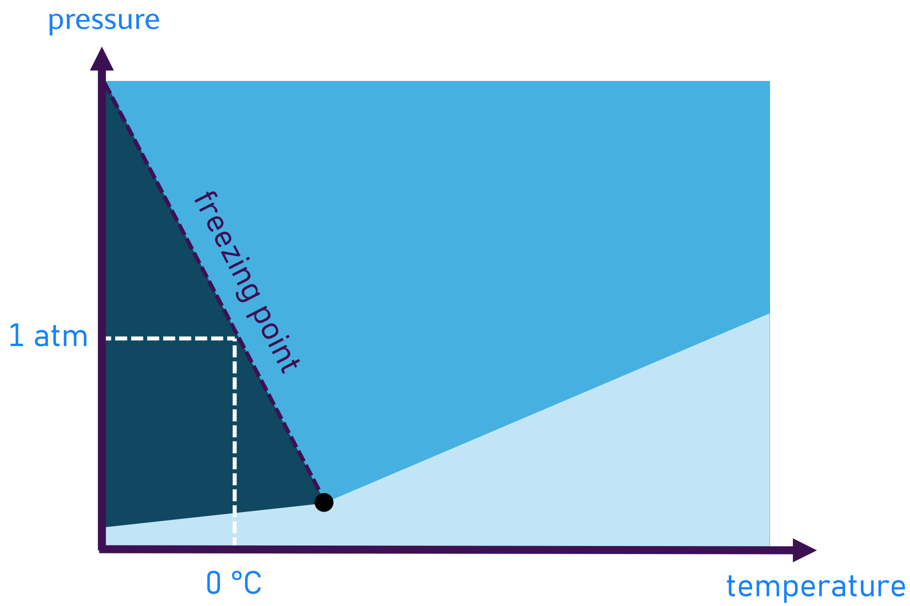
Why does freezing cause damage?

- ~~Thermal expansion? ice has a volume ~9% greater than that of liquid water~~

- Freeze-thaw weathering? repeated expansion and contraction = damage

- Microscale damage? cells burst when frozen and their membranes are permanently destroyed

# How do deformable porous media freeze?




Pressure is raised inside the pores by capillarity, so the *ice-entry temperature* is modified by the Gibbs-Thompson relation

(neither actually derived this...)

$$T_{IE} = T_m \left[1 - \frac{\gamma\kappa}{\rho_{\text{ice}}\mathcal{L}}\right]$$

surface tension and average pore curvature

equilibrium freezing temperature (~273 K)

specific latent heat of fusion

At the boundary, temperature given by Clausius-Clapeyron relation:

(...ditto)

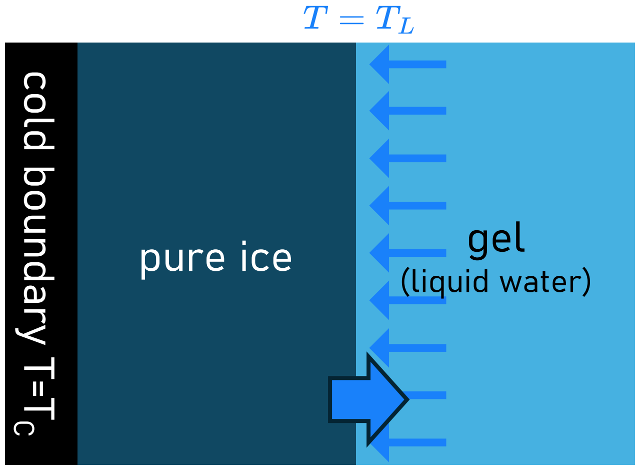
$$\rho_{\text{water}}\mathcal{L}\frac{T_L - T_m}{T_m} = p_{\text{gel}} - p_{\text{atm}} = p_{\text{gel}} + \boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n}$$

$$\mathcal{L}\frac{T_L - T_m}{T_m} = \frac{\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n} + p_{\text{atm}}}{\rho_{\text{ice}}} = \frac{p_{\text{gel}} - p_{\text{atm}}}{\rho_{\text{water}}}$$

$$= -\Pi(\phi)$$

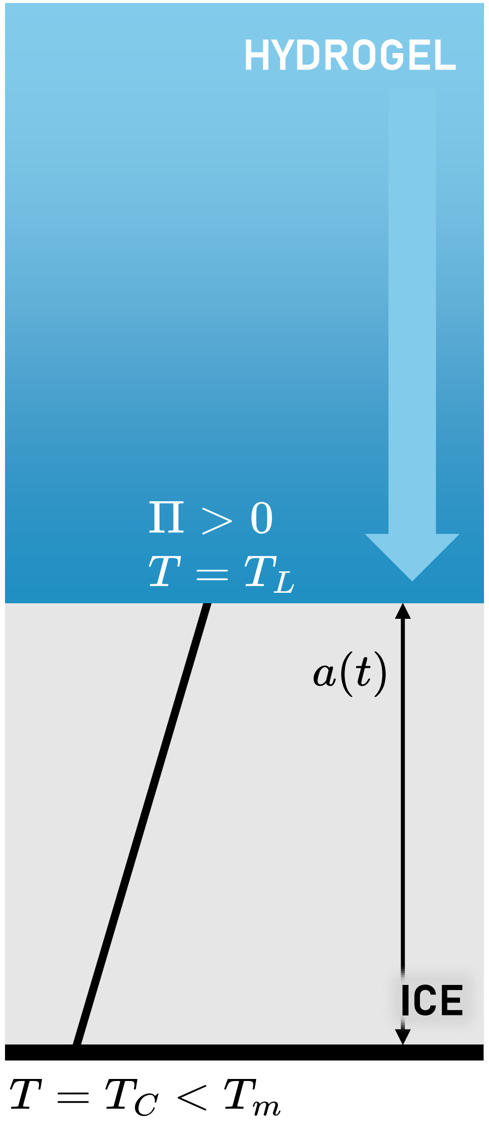
equals (-) bulk pressure

assume no overburden stress $\boldsymbol{n}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{n} = -p_{\text{atm}}$




Worster (2000) *in* Perspectives in Fluid Dynamics *Cambridge Uni. Press*; Style *et al.* (2023) *J. Glaciol.* ***69***:*1091-1096*

# How does ice grow from a hydrogel?




Clausius-Clapeyron relation: temperature depends on how dry the gel is

$$T_L = T_m \left[ 1 - \frac{\Pi(\phi)}{\rho_{\text{water}} \mathcal{L}} \right]$$

**BC on polymer fraction**

Temperature sets the amount of drying

**BC on temperature**

How dry the gel is sets the temperature

Stefan condition: growing ice uses up energy

$$\rho_{\text{ice}} \mathcal{L} \frac{\mathrm{d}a}{\mathrm{d}t} = - \left[ \mathcal{K} \frac{\partial T}{\partial z} \right]_-^+$$

Quasi-steady thermal problem implies $T$ is linear (in ice),

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathcal{K}}{\rho_{\text{ice}} \mathcal{L}} \frac{(T_m - T_C) - T_m \Pi(\phi)/\rho_{\text{water}} \mathcal{L}}{a(t)}$$

Mass conservation: to form ice, water must be drawn from the hydrogel

$$\rho_{\text{ice}} \frac{\mathrm{d}a}{\mathrm{d}t} = -\rho_{\text{water}} \boldsymbol{u} \cdot \boldsymbol{n} = \frac{\rho_{\text{water}} k}{\mu_l} \frac{\partial p}{\partial z}$$

Darcy's law $u = -(k/\mu_l)\partial p/\partial z$

# How does ice grow from a hydrogel?

## The thermal problem

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial z^2} \quad \begin{cases} \text{in the ice } 0 < z < a(t) \\ \text{in the gel } a(t) < z < h \end{cases}$$

$T = T_C$ at $z = 0$

$\partial T / \partial z = 0$ at $z = h$

**whilst at the interface $z = a(t)$,**

$$T = T_m \left[1 - \Pi(\phi)/\rho_{\text{water}}\mathcal{L}\right]$$

$$\rho_{\text{ice}}\mathcal{L}\frac{\mathrm{d}a}{\mathrm{d}t} = -\left[\mathcal{K}\frac{\partial T}{\partial z}\right]_-^+$$

## The gel problem

JJW & Worster Proc. Roy. Soc. A 481:20240721 (2025)

To describe the response of a gel, there are three material parameters:

| $\Pi(\phi)$ | $\mu_s(\phi)$ | $k(\phi)$ |
|---|---|---|
| osmotic pressure | shear modulus | permeability |

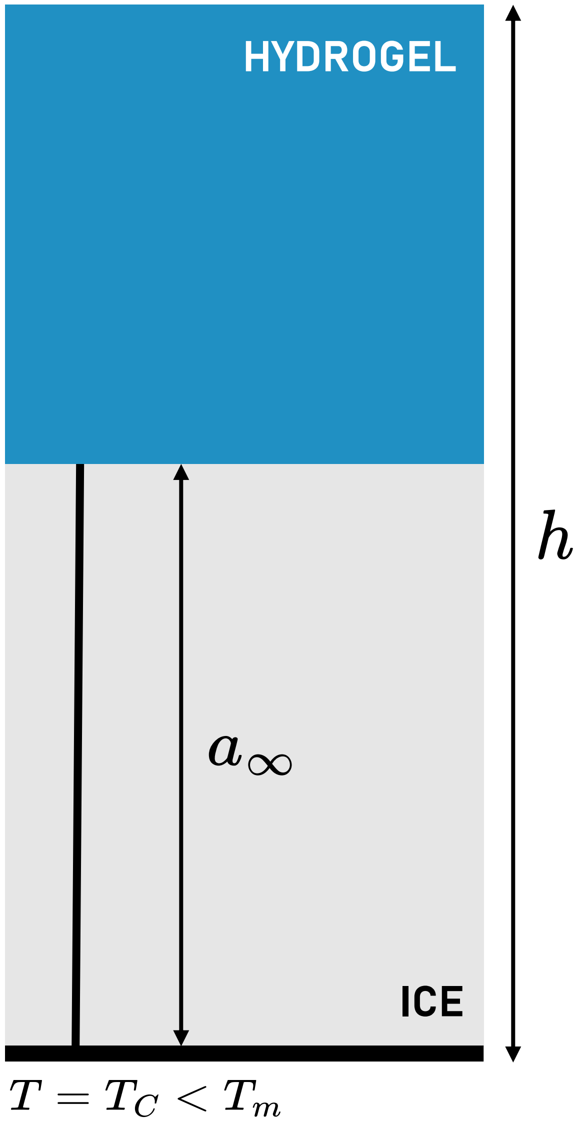
$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial z}\left[D(\phi)\frac{\partial \phi}{\partial z}\right] \qquad \frac{\partial \phi}{\partial z} = 0 \qquad \Pi(\phi) = \rho_{\text{water}}\mathcal{L}(T_m - T_L)$$

in the gel $a(t) < z < h$ — at $z = h$ — at $z = a(t)$

Growth rate of ice governed by mass balance at the interface,

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{D(\phi)}{\phi}\frac{\partial \phi}{\partial z}$$

# The steady state




gel dries → freezing temperature drops → $T_L = T_C$ → freezing stops

*In this steady state…*

- Polymer fraction is uniform (otherwise, flow from wet to dry)
- Temperature is uniformly equal to the liquidus value

$$\Pi\left(\frac{h\phi_0}{h-a}\right) = \rho_{\text{water}}\mathcal{L}(T_m - T_C)$$

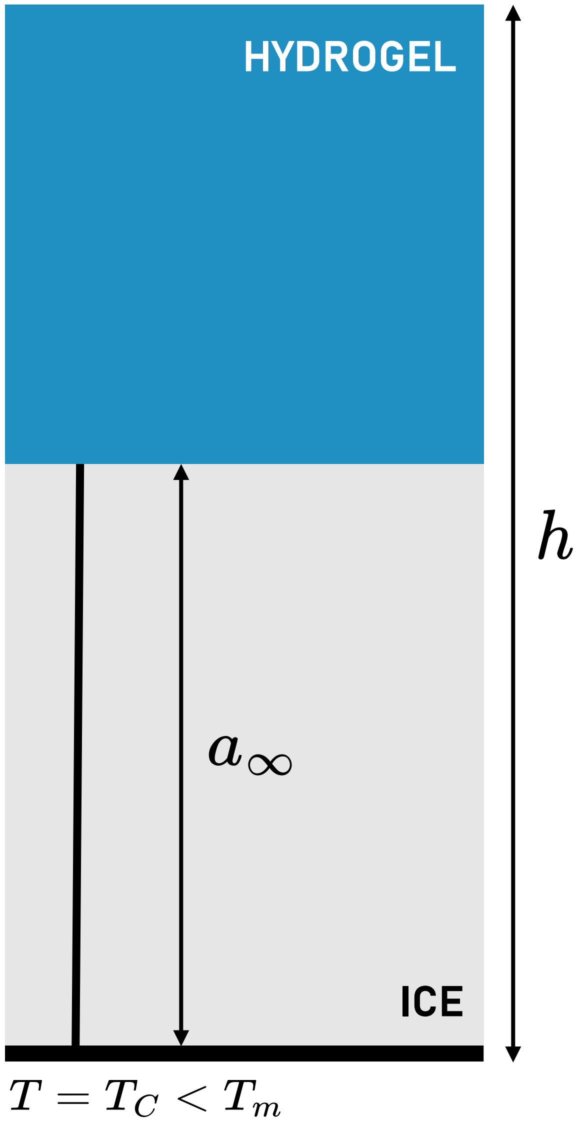
New polymer fraction comes from mass conservation *(swollen value $\phi_0$)* $\int_a^h \phi \,\mathrm{d}x \equiv \phi_0 h$

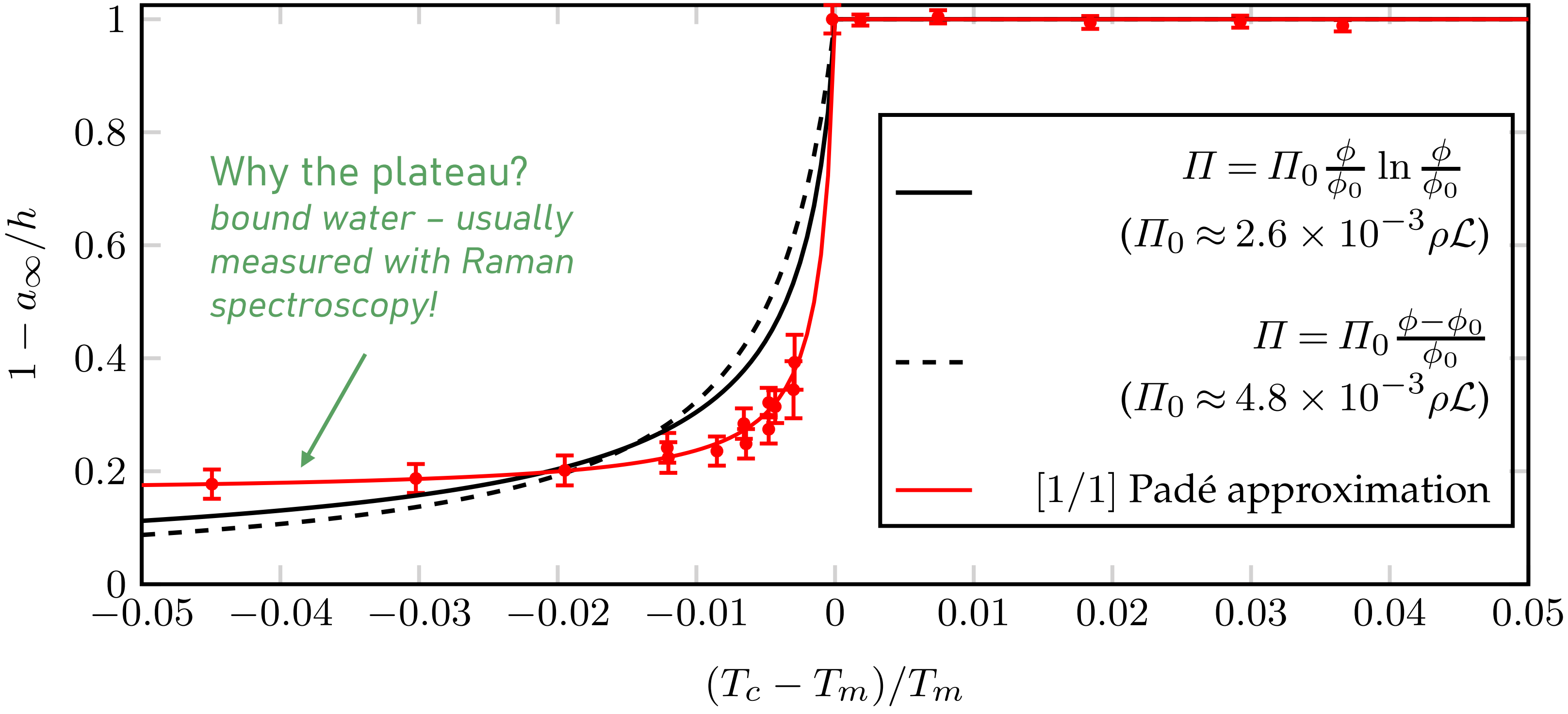
This is the basis for **Gel-freezing osmometry (GelFrO)**

**Feng et al.** J. Mech. Phys. Solids **201**:106166 (2025)

# The steady state

data from **Feng *et al.*** J. Mech. Phys. Solids **201**:106166 (2025)



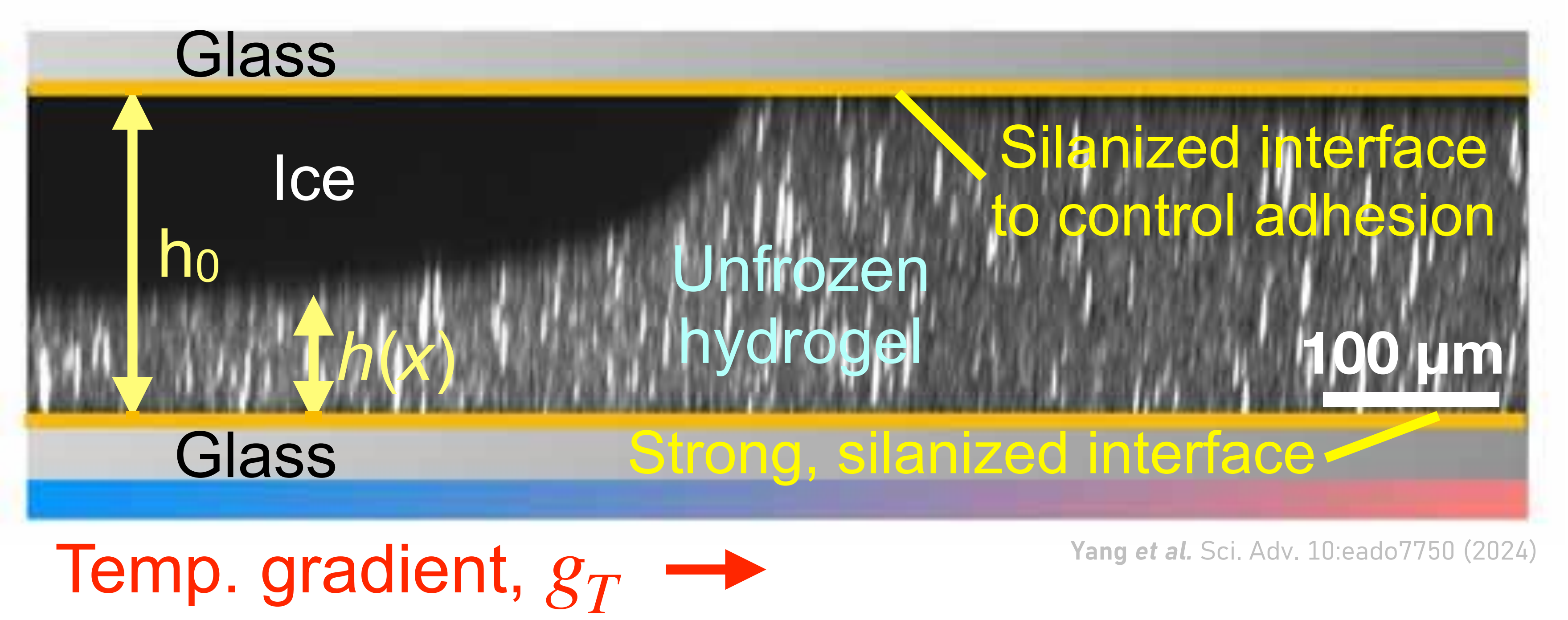




$$\Pi(\phi) = \frac{10^{-3} \rho \mathcal{L}}{\phi_0} \frac{\phi - \phi_0}{1 - \phi/(6.6\phi_0)}$$

**First result:** freezing gels lets us probe their internal structure, including some microscopic properties that are experimentally hard to find!

# How does ice lens growth damage gels?

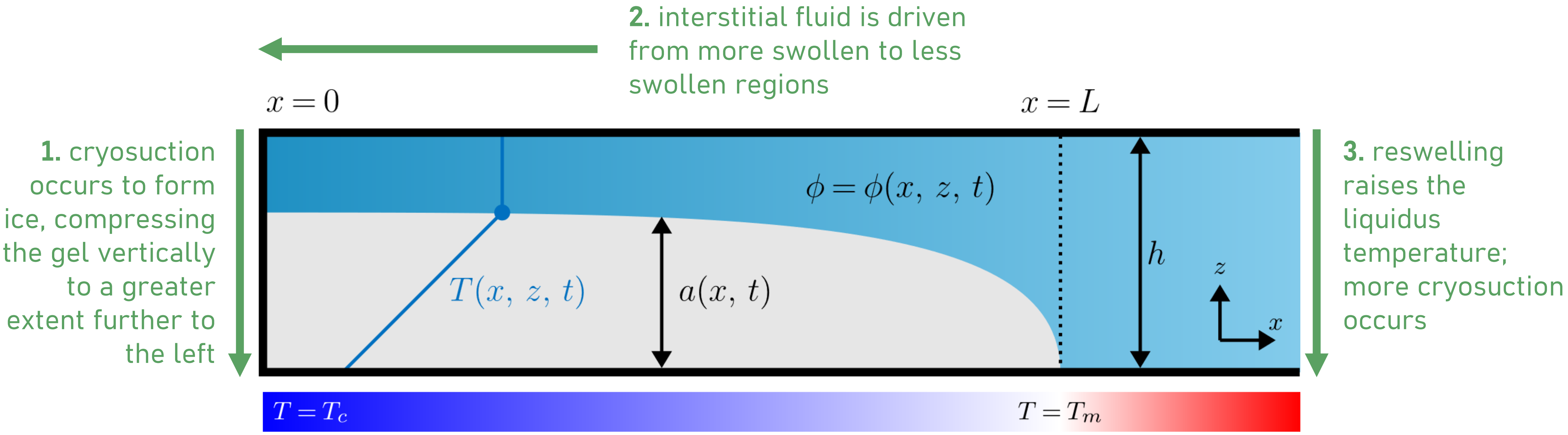
Freezing leads to stress buildup in the dried gel that remains; in our 1D example, this stress is uniform (eventually) through the gel. In 2D, however, the picture is more complicated




**Yang *et al.*** Sci. Adv. 10:eado7750 (2024)

# Forming ice 'lenses'

Freezing leads to stress buildup in the dried gel that remains. In our 1D example, this stress is eventually uniform. In 2D, the picture is more complicated:




This feedback cycle only breaks when reswelling can't occur any longer. What's missing?

drying (↖) → osmotic stress (↖) → pore pressure (↗) → flow (←)

**OR** drying (↖) → elastic stress (↖) → pore pressure (↘) → flow (→) ?

# Modelling displacement

Gradients in pore pressure balance those in osmotic pressures **and** deviatoric (shearing) stress

$\nabla p + \nabla \Pi = 2\nabla \cdot [\mu_s(\phi)\boldsymbol{\epsilon}]$ **+** slenderness *h/L ≪ 1* **+** displacement from equilibrium ***ξ*** *= (ξ, η)*

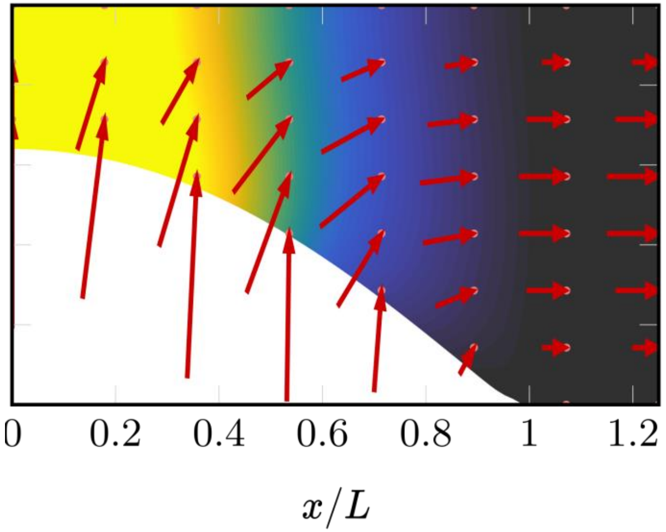
$$\nabla^4 \boldsymbol{\xi} = -3\nabla\nabla^2(\phi/\phi_0)^{1/3}$$

JJW & Worster J. Fluid Mech. 960:A38 (2023)

$$\frac{\partial \phi}{\partial t} + \left(\frac{\phi}{\phi_0}\right)^{-1/2} \frac{\partial \xi}{\partial t}\frac{\partial \phi}{\partial x} + \left(\frac{\phi}{\phi_0}\right)^{-1/2} \frac{\partial \eta}{\partial t}\frac{\partial \phi}{\partial z} = \frac{k(\phi)}{\mu_l}\frac{\partial}{\partial \phi}\left[\Pi(\phi) + 2\mu_s(\phi)\left(\frac{\phi}{\phi_0}\right)^{1/2}\right]\frac{\partial^2 \phi}{\partial z^2}$$

*P = p + Π*

$$\xi = -\frac{1}{2\mu_s}\frac{\partial P}{\partial x}(h-z)(z-a)$$




couple with a quasi-steady temperature profile (heat diffuses faster than water)

make a choice on displacement BCs

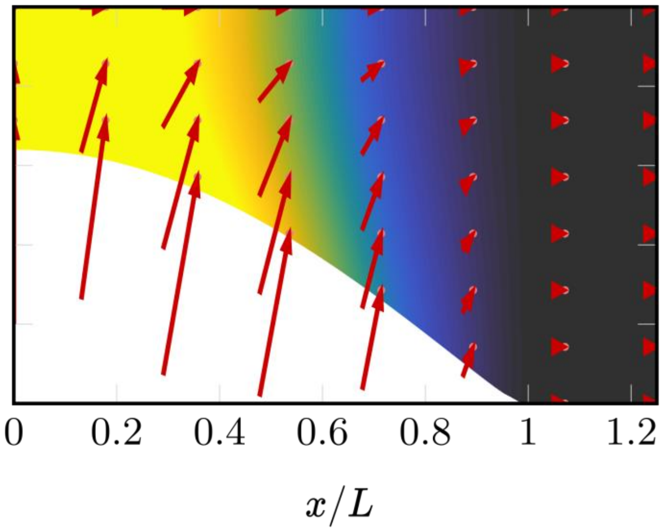
**← NO-SLIP**

*parabolic horizontal displacement*

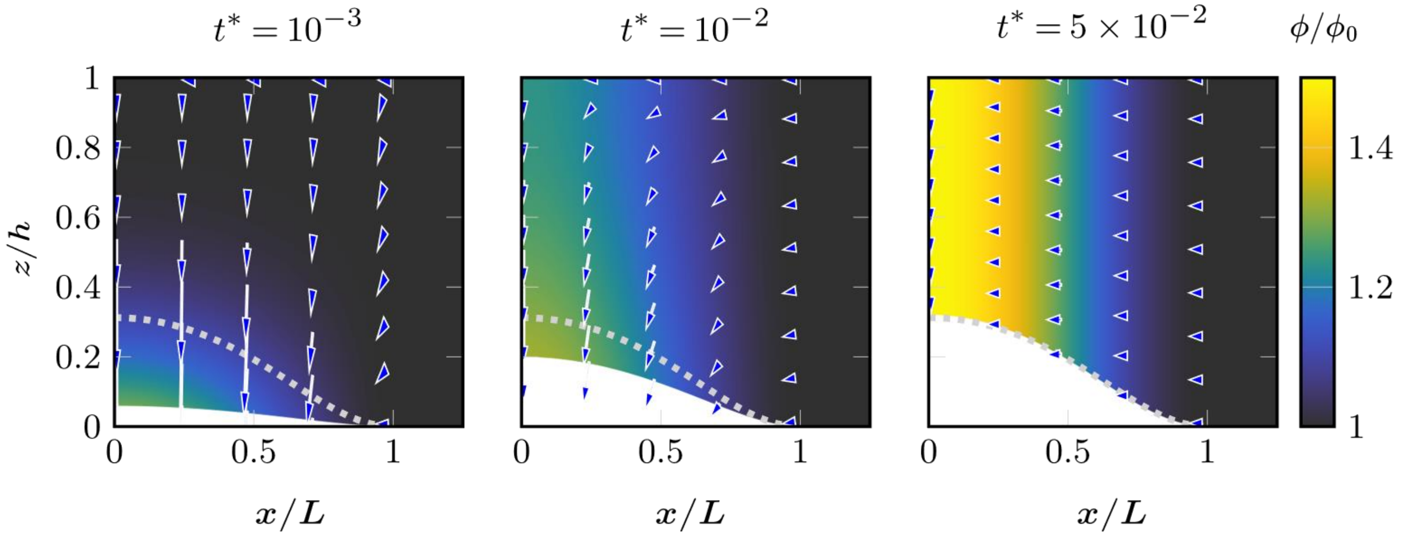
**FREE-SLIP →**

*stretched out uniform horizontal displacement*

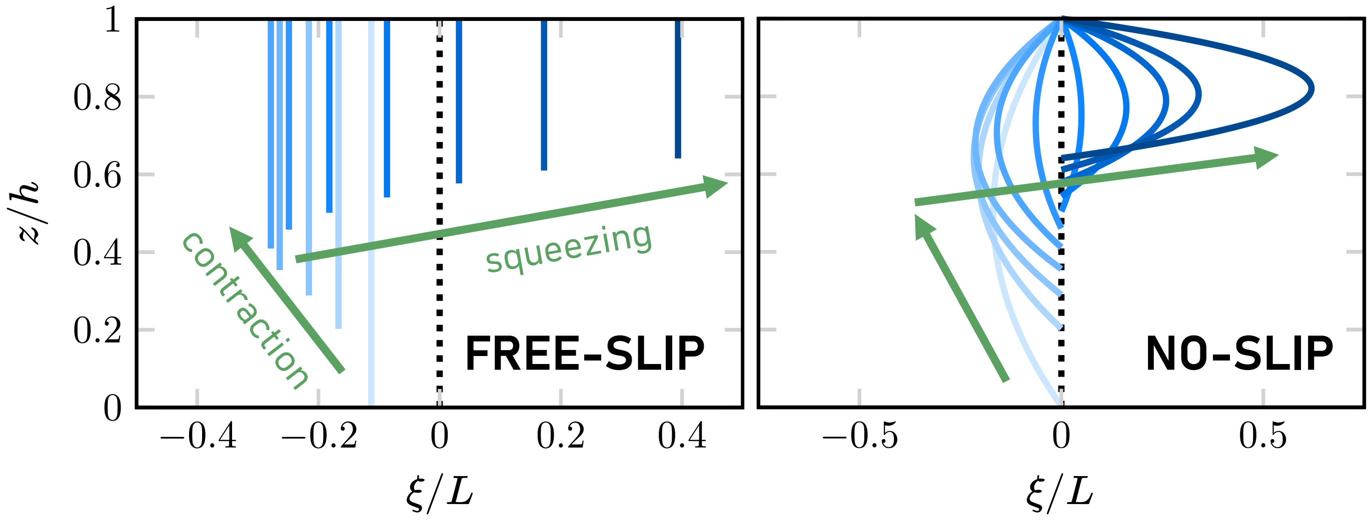
$$\xi = \int_0^x \left\{\frac{a}{h-a} - \frac{2}{h-a}\int_a^h \left[(\phi/\phi_0)^{1/2} - 1\right]\mathrm{d}z'\right\}\mathrm{d}x'$$



# Dynamics of lens growth




↑ time scaled on poroelastic timescale, free-slip BCs; **interstitial fluid velocity shown as blue arrows**




Two phases of gel deformation:

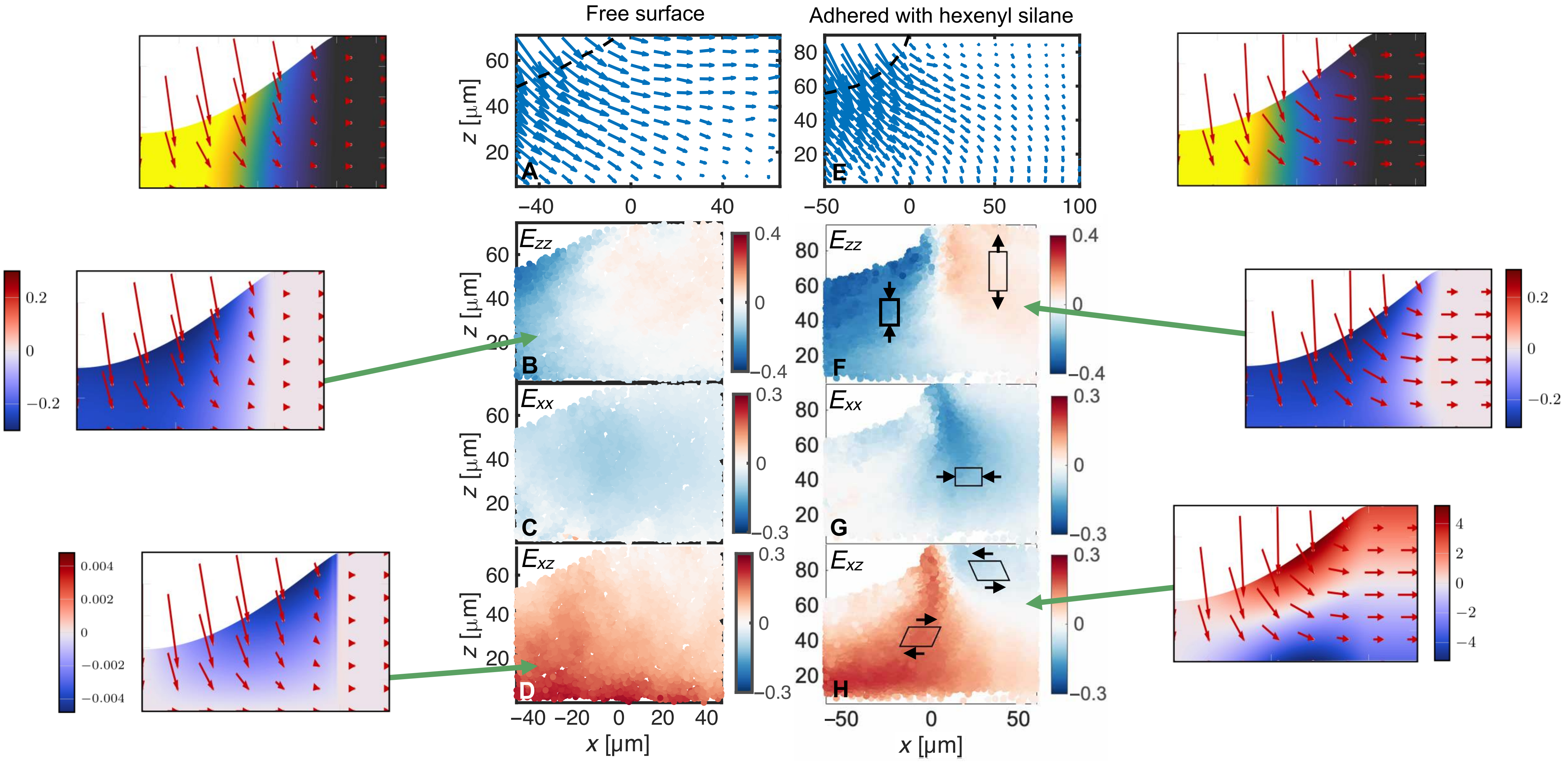
**Contraction** when the gel deswells, driving fluid to the ice and shrinking back in response

**Squeezing** when the growing ice compresses the gel and 'extrudes' it horizontally to the right

Eventually, deviatoric stresses exactly balance osmotic pressure gradients which result from

$$\Pi(\phi_\infty) = \frac{\rho\mathcal{L}}{2}\left(1 - \frac{T_C}{T_m}\right)\left(1 + \cos\frac{\pi x}{L}\right)$$

# Stresses and strains



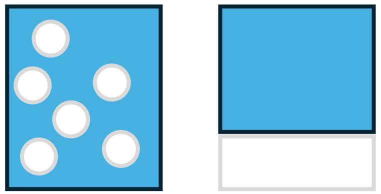

central figure from **Yang et al.** Sci. Adv. 10:eado7750 (2024)

**MODE I** **MODE II**

# Understanding and controlling damage

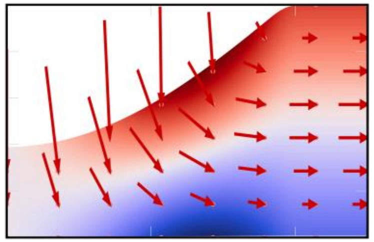
How can we minimise damage, then, to soft materials when freezing them?

- **Change the temperature:**
  dependent on whether we want to freeze water *in situ* or preserve cell structures, choose a temperature either side of the ice-entry value



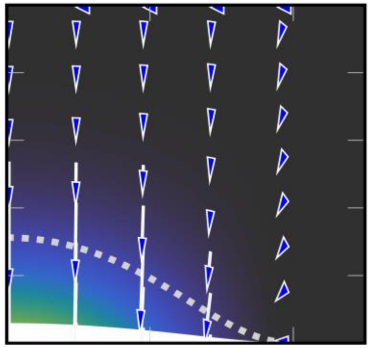
- **Change the confinement:**
  materials bound to stiff substrates build up more damage

  this is key in transplant organs, which should be detached from stiffer materials such as tendons to preserve them better



- **Change the rate:**
  suction can cause damage, and our model quantifies the interstitial flow velocities as a function of undercooling

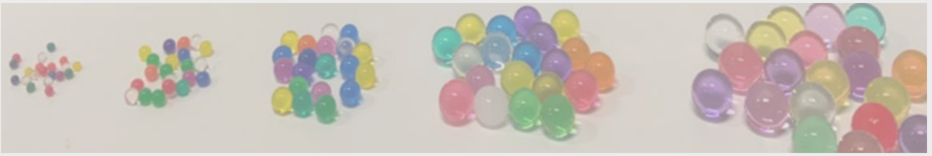
$$a \approx \sqrt{\frac{\mathcal{K}}{\rho \mathcal{L}} (T_m - T_C) \left(1 + \cos \frac{\pi x}{L}\right) t}$$



# Overview

## 1. LENS modelling of gels

Osmosis · elastic stresses · equilibrium swelling · transport of water · shape change in swelling/drying · behaviour at interfaces · characterising a gel

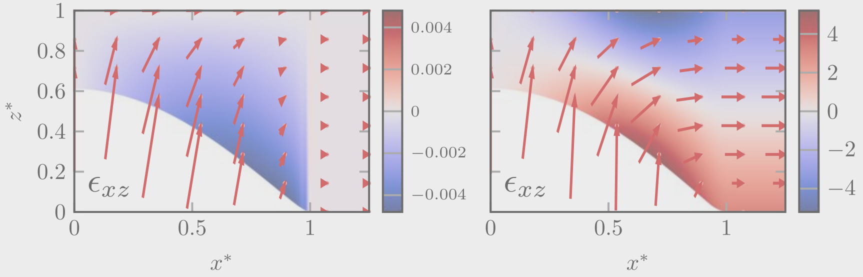


*Temperature (representative values)*

270 K · 300 K · 310 K

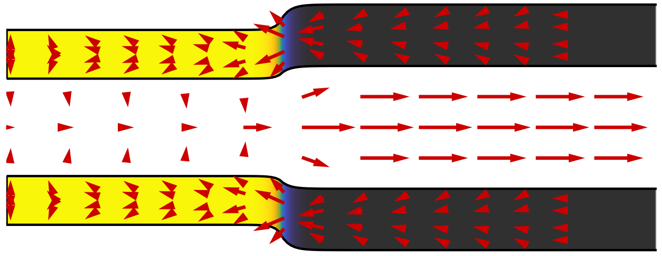
## 2. Freezing gels at low*(-ish)* temperatures

Formation of pure ice · cryosuction · applying model to GelFrO · a 2D model · stress buildup



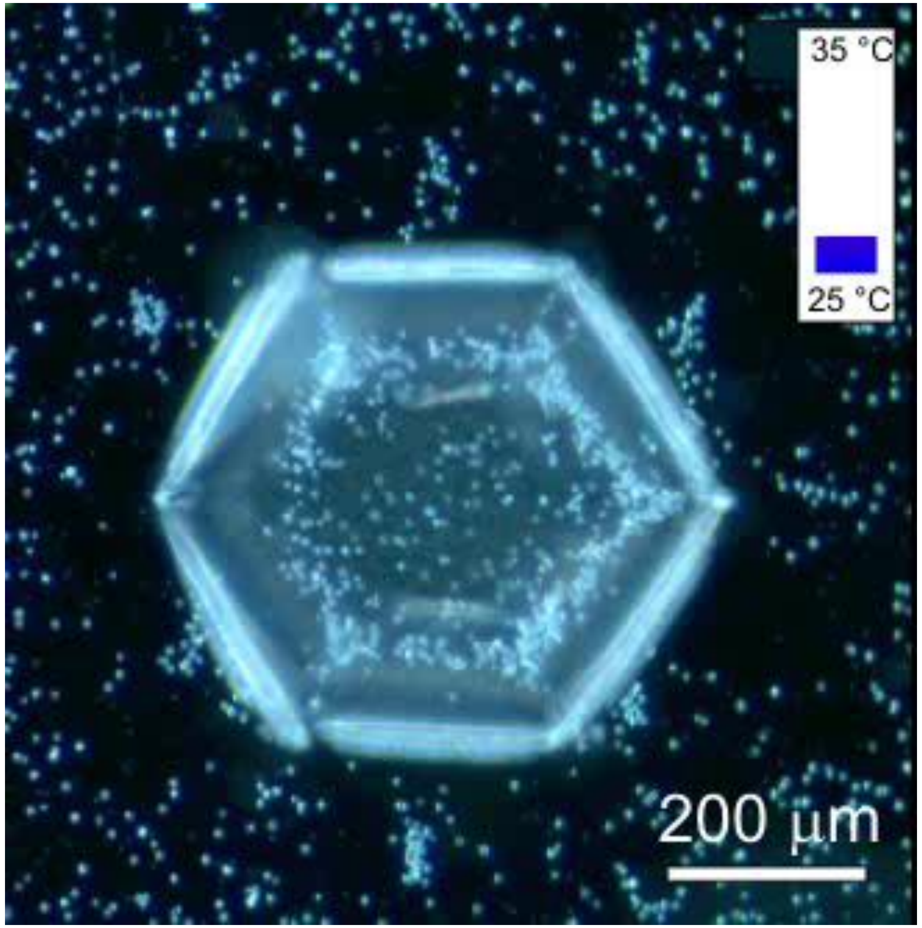

## 3. Building with thermo-responsive gels

Heating effects on swelling · heat transfer in gels · building pumps with collapsing tubes



# Thermo-responsive gels

Some gels, like many based on poly(N-isopropylacrylamide) (pNIPAM) undergo a transition at a critical temperature called the **Lower Critical Solution Temperature (LCST)**




Qualitatively, it appears that there is a new equilibrium (dry) polymer state above the LCST, with transition between the two states slow, mediated by diffusion of water

$$\phi_0 = \begin{cases} \phi_{00} & T \leq T_C \\ \phi_{0\infty} & T > T_C \end{cases}$$

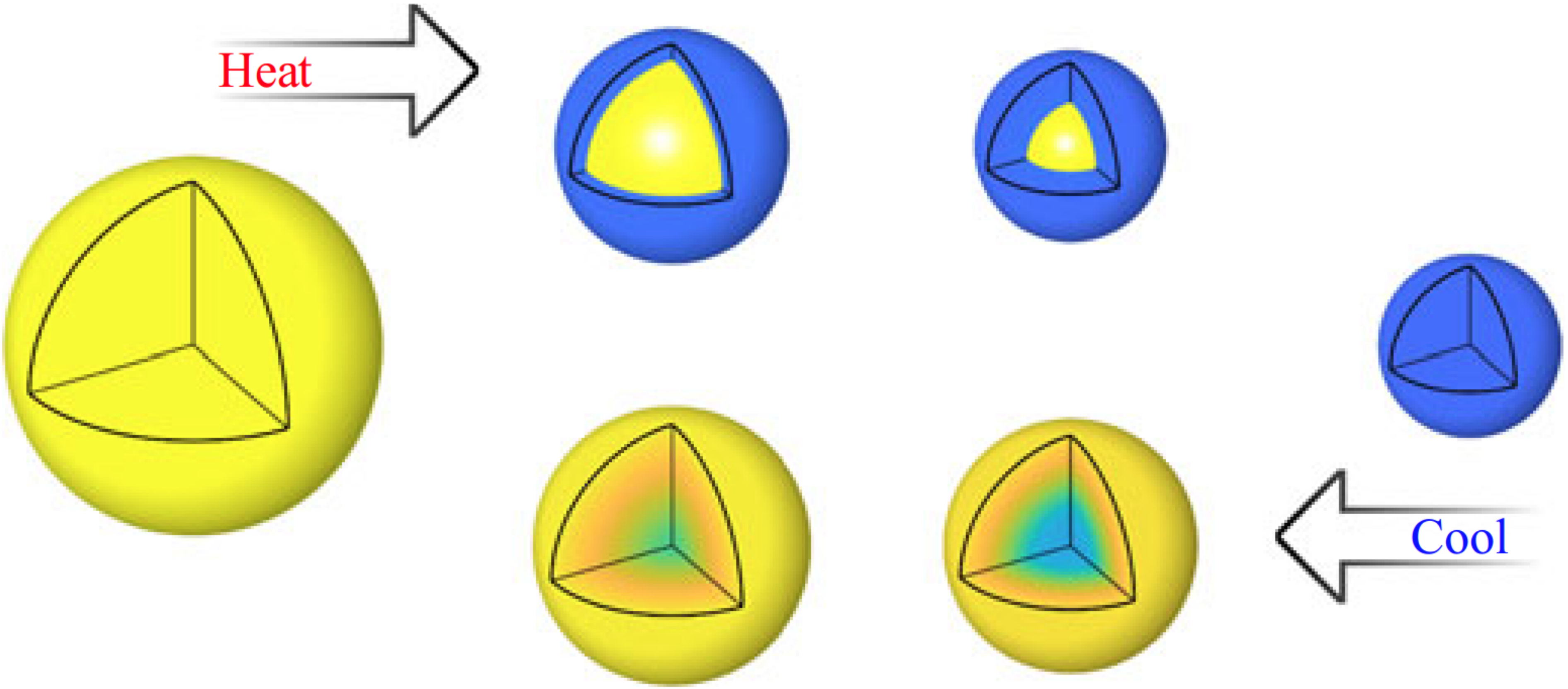
$$\mathcal{W} = \frac{k_B T}{2\Omega_p}\left[\mathrm{tr}\left(\mathbf{F_d}\,\mathbf{F_d^T}\right) - 3 + 2\log\phi\right] + \frac{k_B T}{\Omega_f}\left[\frac{1-\phi}{\phi}\log(1-\phi) + \chi(\phi, T)(1-\phi)\right]$$

Solving the implicit relation $\left.\frac{\partial \mathcal{W}}{\partial \phi}\right|_{\phi=\phi_0} = 0$ gives the equilibrium polymer fraction as a function of temperature if we know the interaction parameter's value

Stoychev et al. (2011) Soft Matter **7**:3277-3279

# Thermo-responsive gels

Take a simple functional form $\chi(\phi, T) = A_0 + A_1 T + (B_0 + B_1 T)\phi + \mathcal{O}(T^2, \phi^2)$

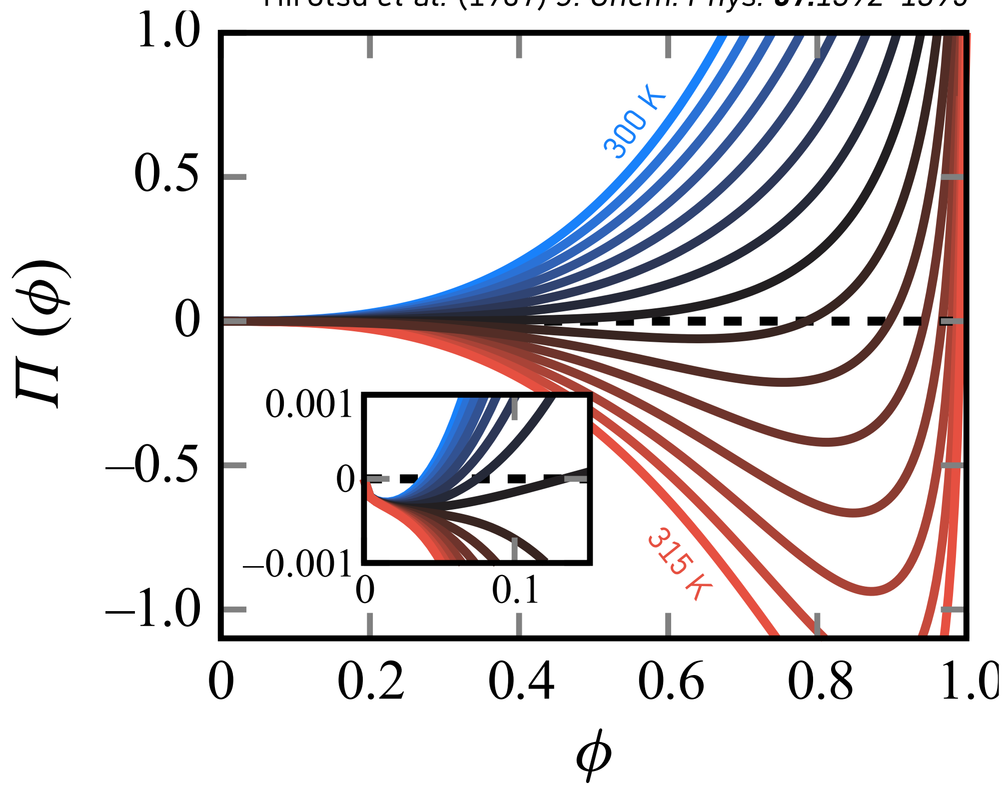



…we're skipping out some significant and important behaviour here, however.

Butler & Montenegro-Johnson (2022) *J. Fluid Mech.* ***947***:*A11*

# Thermo-responsive gels

Hirotsu *et al.* (1987) *J. Chem. Phys.* ***87****:1392–1395*



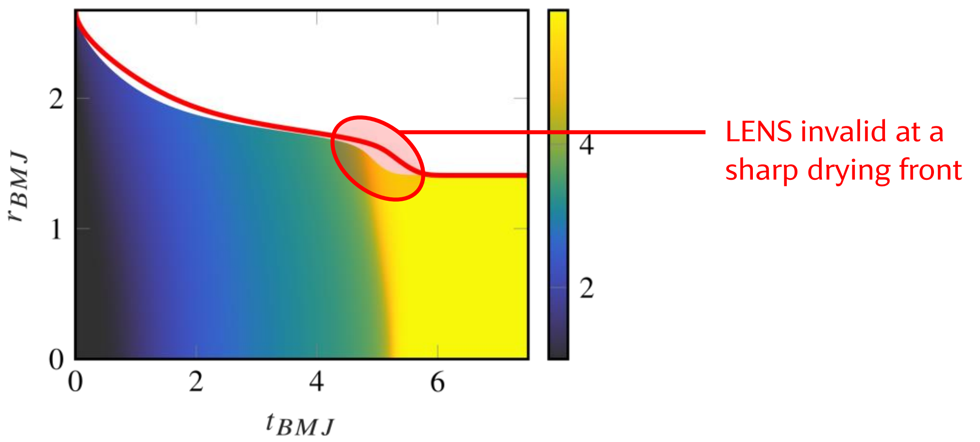

If the same parameter set is used to generate a generalised osmotic pressure (in the LENS formalism), we see the same rapid switch in equilibrium values as the temperature increases

**Phenomenologically**, it suffices to take

$$\Pi(\phi) = \Pi_0 \frac{\phi - \phi_0(T)}{\phi_0(T)}$$

**Big question:** Does this work?

motivating our choice $\phi_0 = \begin{cases} \phi_{00} & T \leq T_C \\ \phi_{0\infty} & T > T_C \end{cases}$




JJW & Montenegro-Johnson (2025) *J. Fluid Mech.* ***1009****:A38*

# Heat transfer in thermo-responsive gels

1. **External supply of heat**
2. **Heat generation due to viscous flows**
3. **Heat transfer by advection**
4. **Heat transfer by diffusion**
5. **Energy used in swelling or drying**

*full derivation in JJW & Montenegro-Johnson (2025)*

External heating Thermal diffusivity Permeability

$$\frac{\partial T}{\partial t} + \boldsymbol{q} \cdot \boldsymbol{\nabla} T = \frac{R}{\rho c} + \kappa \nabla^2 T + \frac{k(\phi)}{\rho c \mu_l} |\boldsymbol{\nabla} p|^2 + \frac{1}{\phi}\left(\frac{\Pi(\phi)}{\rho c} + T\right)\frac{\mathrm{d}\phi}{\mathrm{d}t}$$

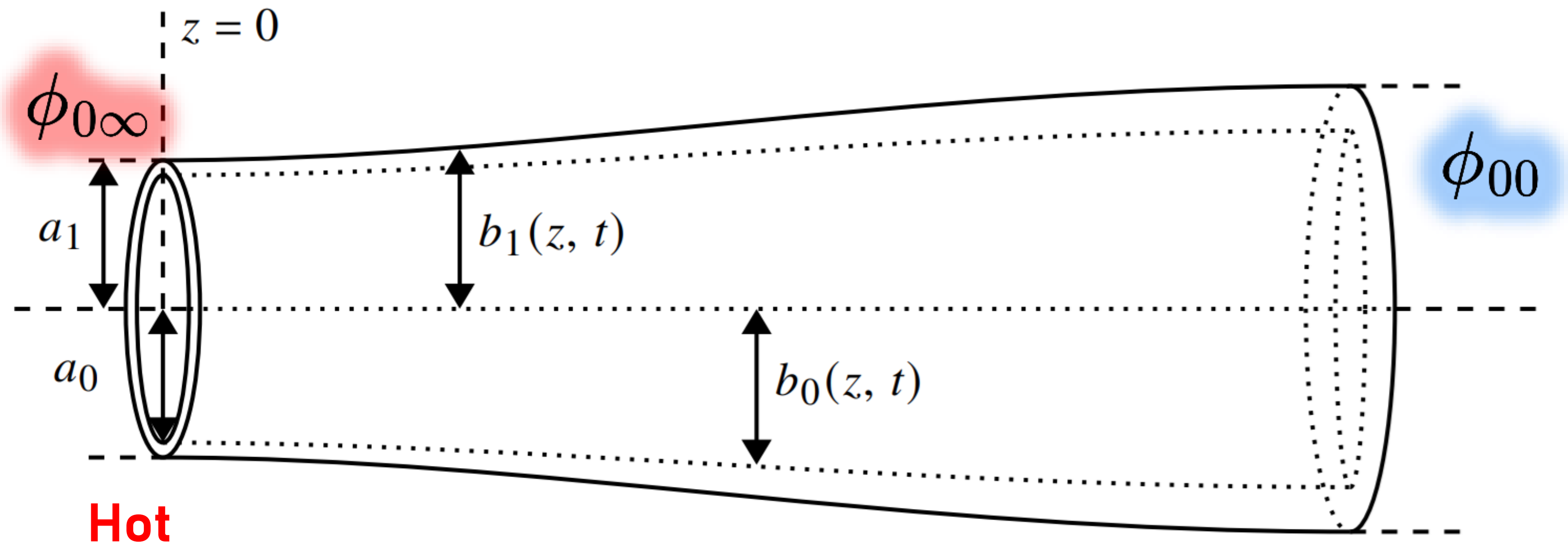
Density Specific heat capacity Fluid viscosity

Usually, however, reconfiguration is 'slow' on the timescale of heat transfer by diffusion (Lewis number –thermal diffusivity over compositional diffusivity – is large), and so we can approximate

$$\frac{\partial T}{\partial t} \approx \kappa \nabla^2 T$$

JJW & Montenegro-Johnson (2025) *J. Fluid Mech.* ***1009***:*A38*

# Tubes of responsive gel




**Hot**
Shrunken tube, narrow lumen
$T = T_C + \Delta T$

**Cold** $T \to T_C - \Delta T$
Relaxed tube, wide lumen

1. How does a heat pulse travel (symmetrically) outwards in time?

2. What happens to the shape of the tube as the pulse passes?

3. Where does the water go? How much is driven out radially, squeezed through the lumen, or transported along the gel?

JJW & Montenegro-Johnson (2025) *J. Fluid Mech.* ***1009***:*A38*

# Heat transfer problem

**1. is easy** *(if we 'spherical cow' the problem a little…)*

The thermal diffusivity of pNIPAM gels is close to that of water, so we can treat the heat transfer problem as occurring in a single infinite domain with only variation in the *z* direction

$\kappa_{\mathrm{gel}} \approx 1.8 \times 10^{-7}\,\mathrm{m^2 s^{-1}}$

Tél *et al.* (2014) *Int. J. Therm. Sci.* ***85***:47-53

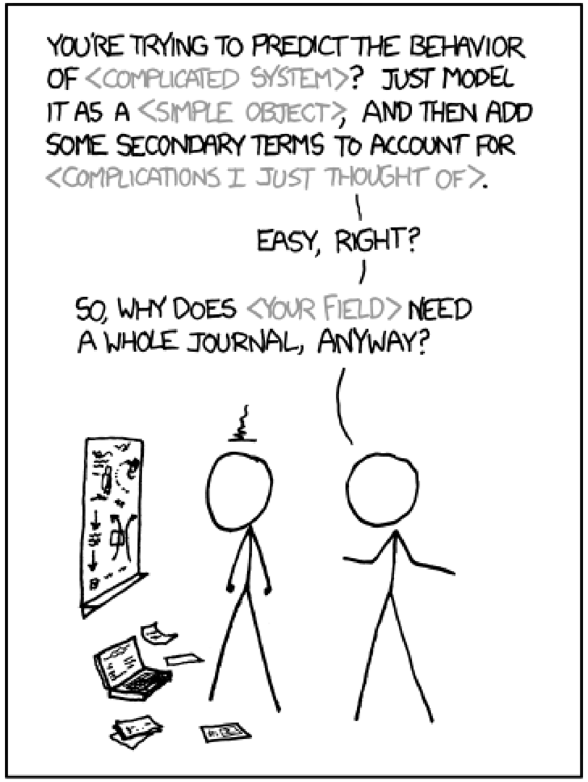
$\kappa_{\mathrm{water}} \approx 1.43 \times 10^{-7}\,\mathrm{m^2 s}$

*at 1atm and 298 K*

$$T - T_C = \Delta T \left[ 2\,\mathrm{erfc}\left( \frac{z}{2\sqrt{\kappa t}} \right) - 1 \right]$$

so there is a 'front' at $Z_C = 2\,\mathrm{erfc}^{-1}\left(\frac{1}{2}\right)\sqrt{\kappa t}$ behind which the gel is deswollen

Randall Munroe *xkcd* ***793***




LIBERAL-ARTS MAJORS MAY BE ANNOYING SOMETIMES, BUT THERE'S *NOTHING* MORE OBNOXIOUS THAN A PHYSICIST FIRST ENCOUNTERING A NEW SUBJECT.

JJW & Montenegro-Johnson (2025) *J. Fluid Mech.* ***1009***:*A38*

# Deformation of the tube

$$\frac{\partial \phi}{\partial t} + \boldsymbol{q}\cdot\boldsymbol{\nabla}\phi = \frac{1}{r}\frac{\partial}{\partial r}\left[rD(\phi,T)\frac{\partial \phi}{\partial r}\right] + \frac{\partial}{\partial z}\left[D(\phi,T)\frac{\partial \phi}{\partial z}\right] \quad \text{with}$$

$$D(\phi,T) = \frac{k}{\mu_l}\left[\frac{\Pi_0(T)\phi}{\phi_0(T)} + \frac{4\mu_s}{3}\left(\frac{\phi}{\phi_{00}}\right)^{1/3}\right].$$

$\phi = \phi_0(T)$ on boundaries

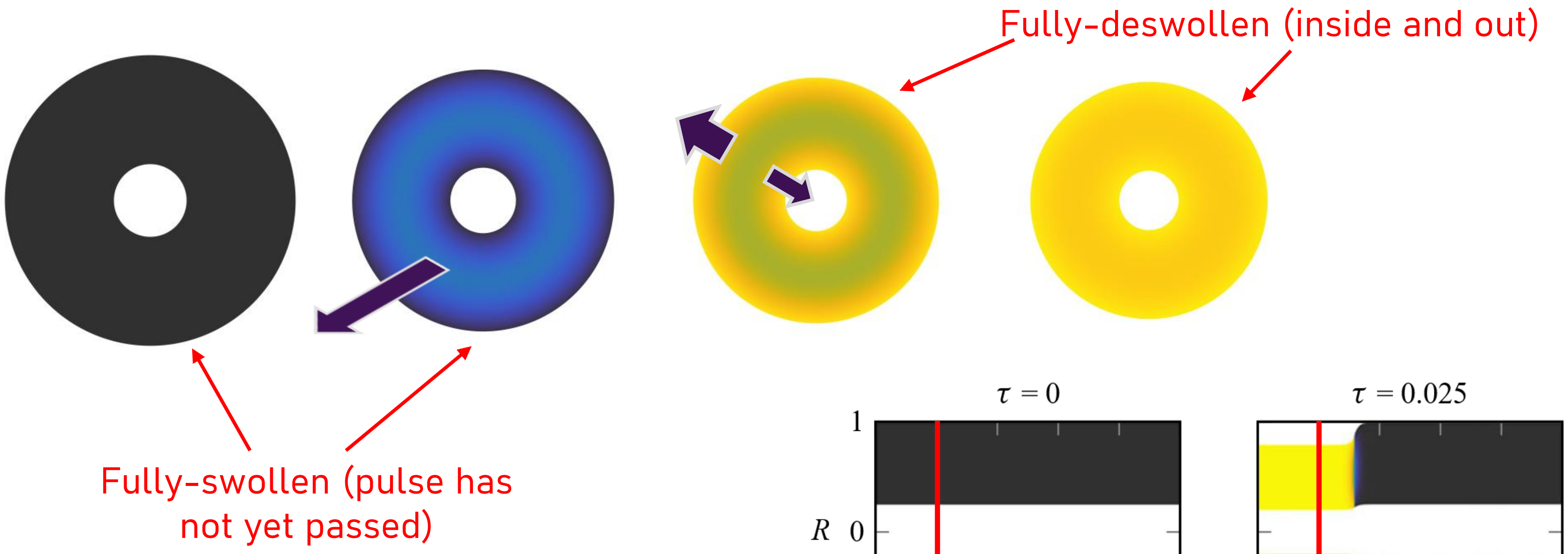
slenderness assumption: aspect ratio small

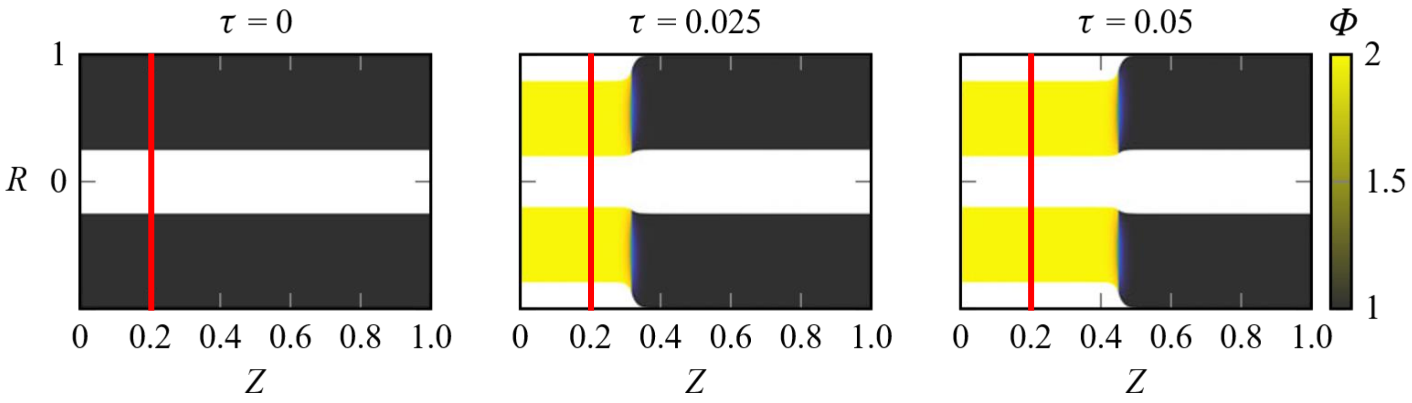
$$\phi = \phi_1(z,\, t) + \varepsilon^2\phi_2(r;\, z,\, t)$$

then separate variables

## 2. Shape change: is a bit harder

To attack this problem, we assume that the tube is long and thin. Balance stresses on its interface with water to find that the gel deswells to its equilibrium value on these surfaces

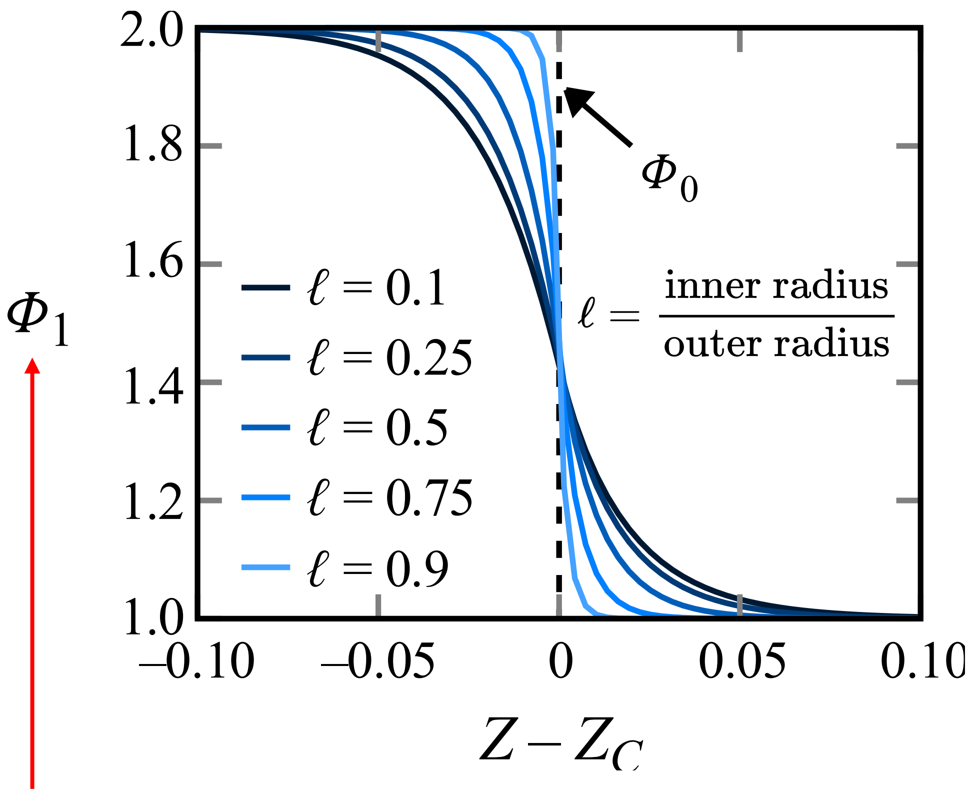



JJW & Montenegro-Johnson (2025) *J. Fluid Mech.* ***1009***:*A38*

# Deformation of the tube

The inside and the outside instantaneously deswell, but the interior takes some time – it's slower for a thicker tube. This leads to 'smoother' profiles for thick tubes.

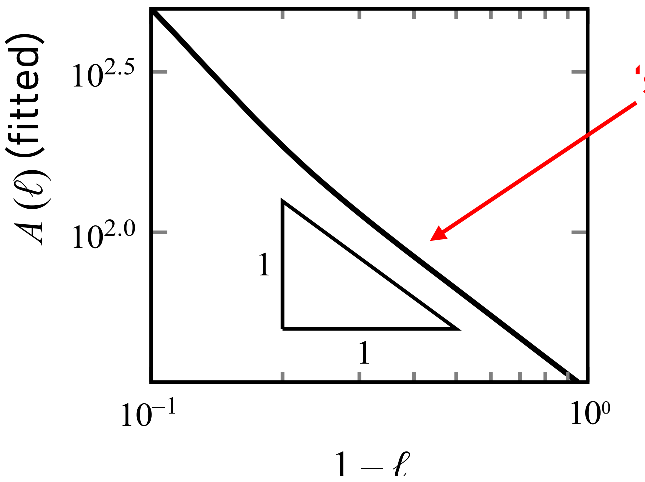



Interior (middle of tube wall) polymer fraction (scaled)

The 'smoothed step' profile suggests that we can nicely approximate the tube with a hyperbolic tangent,

$$\Phi_1 \approx \Phi_\infty - \frac{\Phi_\infty - 1}{2}\{1 + \tanh\left[A(\ell)(Z - Z_C)\right]\}$$

Dry polymer fraction (scaled)

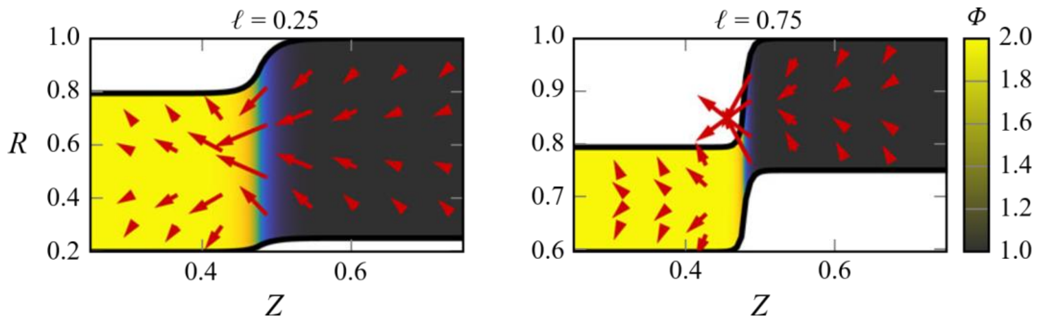



'sharpness' of front scales like 1–$\ell$

This has the corollary of quantifying how much faster the response time is for thinner walls

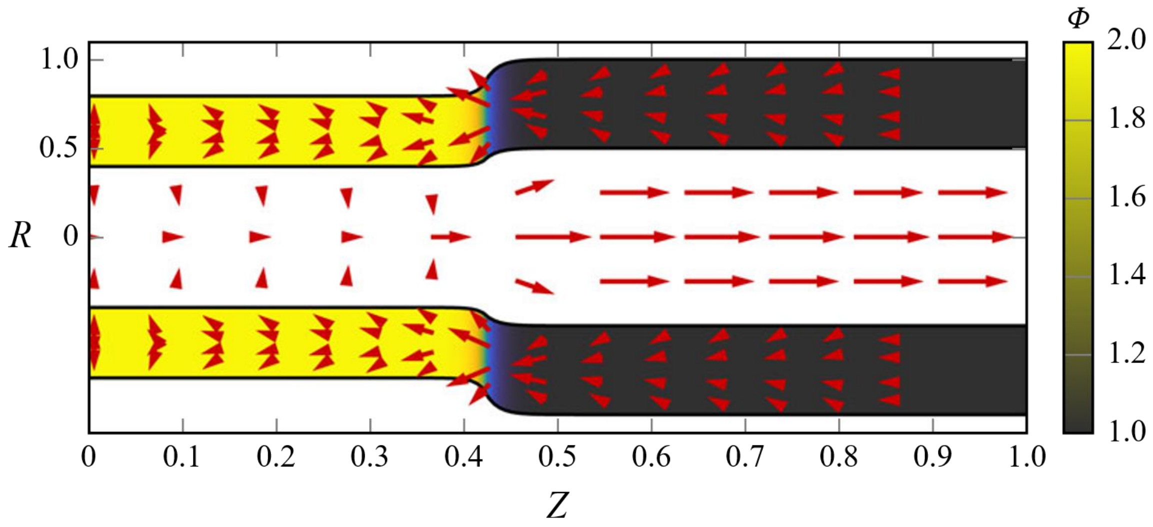
JJW & Montenegro-Johnson (2025) *J. Fluid Mech.* ***1009****:A38*

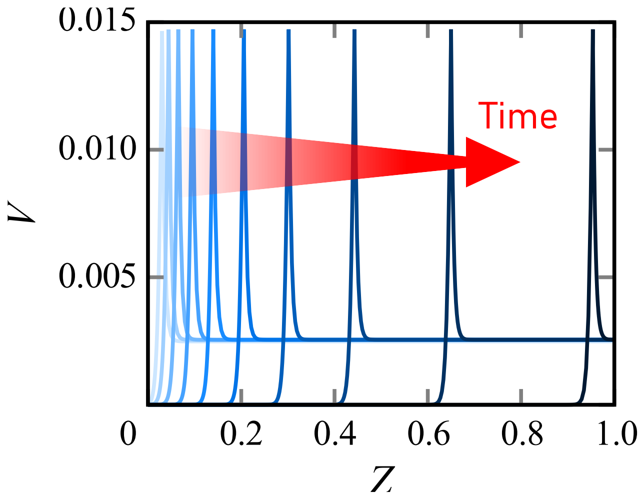
# The fluid pulse




**3. Fluid flows** arise in three places:

- Radial fluxes from the tube walls into the surroundings as they deswell – $u_r \propto \partial\phi/\partial r$
- Axial fluxes through the gel from more swollen to less swollen regions (probably small)
- Axial fluxes arising from conservation of fluid: the tube collapses and squeezes water along its length




JJW & Montenegro-Johnson (2025) *J. Fluid Mech.* ***1009***:*A38*

Applied Mathematics Seminar
University of Nottingham, Thursday 15th January 2026, 15:00

joe.webber@warwick.ac.uk
jwebber.github.io

UK Hydrogels Network

## with thanks to

Tom Montenegro-Johnson
Warwick

Grae Worster
Cambridge

## more details can be found in

**Webber, J. J. & Worster, M. G.** *J. Fluid Mech.* 960:A37 (2023)
**Webber, J. J., Etzold, M. A. & Worster, M. G.** *J. Fluid Mech.* 960:A38 (2023)
**Webber, J. J. & Worster, M. G.** *Phys. Rev. E* 109:044602 (2024)
**Webber, J. J. & Montenegro-Johnson, T. D.** *J. Fluid Mech.* 1009:A38 (2025)
**Webber, J. J. & Worster, M. G.** *Proc. Roy. Soc. A* 481:20240721 (2025)
**Webber, J. J. & Montenegro-Johnson, T. D.** *Phys. Rev. Res.* 7:L032055 (2025)
**Webber, J. J. & Montenegro-Johnson, T. D.** *Phys. Rev. Fluids* 10:100501 (2025)

LEVERHULME TRUST
UNIVERSITY OF WARWICK